

SOME FINITENESS RESULTS FOR GROUPS OF AUTOMORPHISMS OF MANIFOLDS

ALEXANDER KUPERS

ABSTRACT. We prove that in dimension $\neq 4, 5, 7$ the homology and homotopy groups of the classifying space of the topological group of diffeomorphisms of a disk fixing the boundary are finitely generated in each degree. The proof uses homological stability, embedding calculus and the arithmeticity of mapping class groups. From this we deduce similar results for the homeomorphisms of \mathbb{R}^n and various types of automorphisms of 2-connected manifolds.

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1. INTRODUCTION

Inspired by work of Weiss on Pontryagin classes of topological manifolds [Wei15], we use several recent advances in the study of high-dimensional manifolds to prove a structural result about diffeomorphism groups. We prove the classifying spaces of such groups are often “small” in one of the following two algebro-topological senses:

Definition 1.1. Let X be a path-connected space.

- X is said to be of *homologically finite type* if for all $\mathbb{Z}[\pi_1(X)]$ -modules M that are finitely generated as abelian groups, $H_*(X, M)$ is finitely generated in each degree.
- X is said to be of *finite type* if $\pi_1(X)$ is finite and $\pi_i(X)$ is finitely generated for $i \geq 2$.

Being of finite type implies being of homologically finite type, see Lemma 2.15.

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Let $\text{Diff}(-)$ denote the topological group of diffeomorphisms in the C^∞ -topology, $\text{PL}(-)$ the simplicial group of PL-homeomorphisms, $\text{Top}(-)$ the topological group of homeomorphisms in the compact-open topology. A subscript ∂ means we restrict to the subgroup of those automorphisms that are the identity on the boundary, and a superscript $+$ means we restrict to orientation-preserving automorphisms. The following solves Problems 1(b) and 1(d) of Burghlea in [Bur71].

Theorem A. *Let $n \neq 4, 5, 7$, then $B\text{Diff}_\partial(D^n)$ is of finite type.*

Corollary B. *Let $n \neq 4, 5, 7$, then $B\text{Diff}(S^n)$ is of finite type*

Corollary C. *Let $n \neq 4, 5, 7$. Suppose that M is a closed 2-connected oriented smooth manifold of dimension n , then $B\text{Diff}^+(M)$ is of homologically finite type.*

It is convention to denote $\text{PL}(\mathbb{R}^n)$ by $\text{PL}(n)$ and $\text{Top}(\mathbb{R}^n)$ by $\text{Top}(n)$.

Corollary D. *Let $n \neq 4, 5, 7$, then $B\text{Top}(n)$ and $B\text{PL}(n)$ are of finite type.*

Corollary E. *Let $n \neq 4, 5, 7$, then $B\text{Top}(S^n)$ and $B\text{PL}(S^n)$ are of finite type.*

Corollary F. *Let $n \neq 4, 5, 7$. Suppose that M is a closed 2-connected oriented smoothable manifold of dimension n , then $B\text{PL}^+(M)$ and $B\text{Top}^+(M)$ are of homologically finite type.*

For completeness, Propositions 5.20 and 5.22 give similar results for homotopy automorphisms, block automorphisms and the quotient of block automorphisms by automorphisms. This uses mostly classical techniques.

Using the link between diffeomorphisms and algebraic K-theory, we deduce finiteness properties of the spectra $\text{Wh}^{\text{Diff}}(*)$ and $A(*)$ appearing in Waldhausen's algebraic K-theory of spaces [Wal85]. The following result was previously proved by Dwyer [Dwy80].

Corollary G. *We have that $\text{Wh}^{\text{Diff}}(*)$ and $A(*)$ are of finite type.*

The input to our proofs is a collection of deep theorems in manifold theory: the homological stability results of Galatius and Randal-Williams [GRW14a, GRW14b] and Botvinnik and Perlmutter [BP15, Per15], the embedding calculus of Weiss [Wei99, BdBW15], the excision estimates of Goodwillie and Klein [GK15], and the arithmeticity results of Sullivan [Sul77, Tri95].

These are combined with a fiber sequence we call the *Weiss fiber sequence*, as it underlies the work of Weiss in [Wei15]. This fiber sequence expresses the diffeomorphisms of a disk as the difference between the diffeomorphisms of a manifold and certain self-embeddings of that manifold. The setup is as follows: let M be an n -dimensional smooth manifold with non-empty boundary ∂M and an embedded disk $D^{n-1} \subset \partial M$, then $\text{Emb}_{\partial/2}^\cong(M)$ is the space of embeddings $M \hookrightarrow M$ that are the identity on $\partial M \setminus \text{int}(D^{n-1})$ and isotopic fixing $\partial M \setminus \text{int}(D^{n-1})$ to a diffeomorphism that fixes the boundary. In Section 4 we construct a fiber sequence

$$(1) \quad B\text{Diff}_\partial(D^n) \rightarrow B\text{Diff}_\partial(M) \rightarrow B\text{Emb}_{\partial/2}^\cong(M)$$

and show it deloops once to a fiber sequence

$$(2) \quad B\text{Diff}_\partial(M) \rightarrow B\text{Emb}_{\partial/2}^\cong(M) \rightarrow B(B\text{Diff}_\partial(D^n), \mathfrak{t})$$

with \natural denoting boundary connected sum. The advantage of (2) is that the base is 1-connected, so we can use technical input about spaces of homologically and homotopically finite type discussed in Section 2.

The previously mentioned deep theorems imply that there exist manifolds $W_{g,1}$ and H_g , such that the group of the diffeomorphisms of a disk is the only unknown term in (2). Thus we can solve for $B\text{Diff}_\partial(D^n)$. This insight is due to Weiss, and was used by him in [Wei15] to study the rational cohomology of $B\text{Top}(n)$, related to $\text{Diff}_\partial(D^n)$ by smoothing theory. We instead use it to study finiteness properties of $\text{Diff}_\partial(D^n)$.

1.1. Historical remarks. We discuss related results in the literature. This discussion is not complete and does not cover the results used in our argument, e.g. [GRW14b, GRW14a, Wei15, BP15, Per15].

We start with diffeomorphisms of a disk. As often in smooth manifold theory, there is the following trichotomy: (i) *low dimensions* ≤ 3 , (ii) *dimension 4*, (iii) *high dimensions* $n \geq 5$.

- (i) In low dimensions ≤ 3 , the diffeomorphisms of a disk are contractible. For $n = 1$ this is a folklore result, for $n = 2$ this is due to Smale [Sma59b], and for $n = 3$ this is due to Hatcher [Hat83].
- (ii) In dimension 4, nothing is known.
- (iii) In high dimensions ≥ 5 , the homotopy groups of diffeomorphisms of a disk are only understood in low degrees.

The connected components are known: the group $\pi_0(\text{Diff}_\partial(D^n))$ is isomorphic to the group Θ_{n+1} of diffeomorphism classes of homotopy $(n+1)$ -spheres under connected sum [Lev70]. The h -cobordism theorem proves surjectivity and Cerf's theorem proves injectivity [Cer70]. This group is known to be finite abelian and is closely related to the stable homotopy groups of spheres [KM63].

The higher homotopy groups can be determined in a range using the pseudoisotopy stability theorem [Igu88] and algebraic K -theory [Wal85]. In [FH78], Farrell and Hsiang proved that in the so-called concordance stable range $0 < i < \frac{n}{6} - 7$ (improved to $0 < i < \min(\frac{n-7}{2}, \frac{n-4}{3})$ by Igusa) we have that

$$\pi_i(\text{Diff}_\partial(D^n)) \otimes \mathbb{Q} = \begin{cases} 0 & \text{if } n \text{ is even} \\ K_{i+2}(\mathbb{Z}) \otimes \mathbb{Q} & \text{if } n \text{ is odd} \end{cases}$$

The latter is given by \mathbb{Q} if $i \equiv 3 \pmod{4}$ and 0 otherwise [Bor74]. Igusa's work on higher torsion invariants gives an alternative proof [Igu02, Section 6.5].

There are examples of non-zero homotopy groups. We start with rational results. Watanabe's work shows that for many n we have that $\pi_{2n-2}(B\text{Diff}_\partial(D^{2n+1})) \otimes \mathbb{Q} \neq 0$ [Wat09a], and gives lower bounds on the dimension of $\pi_{kn-k}(B\text{Diff}_\partial(D^{2n+1})) \otimes \mathbb{Q}$ for any $k \geq 2$ [Wat09b]. One is tempted to conjecture a relationship between these classes and the embedding calculus employed in this paper.

We continue with torsion results. Novikov found torsion elements in $\pi_i(B\text{Diff}_\partial(D^n))$ [Nov63]. Burghlea and Lashof proved that there is an infinite sequence (p_i, k_i, n_i) with p_i an odd prime and $\lim_{i \rightarrow \infty} n_i/k_i = 0$ such that $\pi_{k_i}(\text{Diff}_\partial(D^{n_i})) \otimes \mathbb{Z}/p_i\mathbb{Z} \neq 0$ [BL74, Section 7], improving on an earlier result by Miller [Mil75]. Work of Crowley and Schick extended this to prove that for $n > 6$, there are infinitely many i such

that $\pi_i(\text{Diff}_\partial(D^n))$ is non-zero, containing an element of order 2 [CS13], which was improved to $n \geq 6$ in [CSS16].

Next we discuss classical results about finiteness properties for automorphism groups of manifolds.

- (i) In low dimensions ≤ 3 , any topological or PL manifold admits a unique smooth structure and for any smooth manifold M (possibly with non-empty boundary) we have

$$\text{Diff}_\partial(M) \simeq \text{PL}_\partial(M) \simeq \text{Top}_\partial(M)$$

In dimension $n = 1$, the only connected manifolds are D^1 and S^1 . The former was discussed above, and $\text{Diff}(S^1) \simeq O(2)$. In dimension $n = 2$, $B\text{Diff}_\partial(\Sigma)$ of homologically finite type for all Σ . We do not know a reference, but will outline two approaches. In the analytic approach one relates $B\text{Diff}_\partial(\Sigma)$ to combinatorially defined spaces using the theory of quadratic differentials or harmonic functions [Str84, Bød06]. In the topological approach, one proves that the connected components of $\text{Diff}_\partial(\Sigma)$ are contractible except in a finite number of exceptional cases, when they have simple homotopy types [Gra73]. One then shows that the mapping class group $\pi_0(\text{Diff}_\partial(\Sigma))$ is of homologically finite type by induction using arc complexes as in [Hat91].

In dimension $n = 3$, using [HL84] the finiteness properties of diffeomorphism groups can be reduced to questions about prime factors and a space related to outer-space. The latter can be studied using the techniques of [HV98]. For the former, we remark that the diffeomorphisms groups of most but not all 3-manifolds are understood. We will not give all relevant references, but will note that the case of Haken manifolds was settled in [Hat76, Iva76], which also shows how to reduce from prime 3-manifolds to atoroidal 3-manifolds. Similarly the case of irreducible M with non-empty boundary was settled by Hatcher and McCullough [HM97]. We also note that Kalliongis and McCullough showed that for many 3-manifolds M , $\pi_1(\text{Diff}(M))$ is not finitely generated [KM96].

- (ii) In dimension 4, the only known results concern mapping class groups of topological 4-manifolds with good fundamental groups. For example, Quinn proved for 1-connected closed M the mapping class group $\pi_0(\text{Top}(M))$ is equal to the group of automorphisms of the intersection form [Qui86]. The same result holds stably, i.e. any two diffeomorphisms inducing the same automorphism of the intersection form become isotopic after taking a connected sum with some number of $S^2 \times S^2$. Ruberman gave examples showing that such stabilizations are necessary [Rub98].
- (iii) In high dimensions ≥ 5 , Farrell and Hsiang also did computations for spheres and aspherical manifolds [FH78]. Burghelea extended these results by proving that for a simply-connected manifold M , the higher rational homotopy groups of the identity component of $\text{Diff}(M)$ are finite-dimensional in the concordance stable range [Bur79]. For the latest results concerning homeomorphisms of aspherical manifolds, see the work of Enkelmann, Lück, Pieper, Ullmann and Winges [ELP⁺16]. Sullivan proved that π_0 is commensurable with an arithmetic group if M is simply-connected [Sul77]. This fails if the manifold is not simply-connected: for $n \geq 5$, $\pi_0(\text{Diff}(T^n))$ is not finitely generated [ABK71, Hat78]. This was generalized to higher homotopy groups of $\text{Diff}(T^n)$

by Hsiang and Sharpe [HS76]. Corollary II.4.6 of [HW73] gives another example of this phenomenon (the statements in [HW73] were corrected by Igusa in the case of non-vanishing Postnikov invariant [Igu84]). Antonelli, Burghlea and Kahn proved that in general the identity component of $\text{Diff}(M)$ does not have the homotopy type of a finite CW-complex [ABK70], which Hirsch and Lawson previously proved for particular manifolds [Hir53, Law72].

These results raise two questions about our results:

Question 1.2. Corollaries C and F have the assumption that M is 2-connected. Is this a necessary hypothesis, or does it suffice that M is simply-connected or has finite fundamental group?

Question 1.3. Are Theorem A and its corollaries true in dimensions 5 and 7?

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2. HOMOLOGICALLY AND HOMOTOPICALLY FINITE TYPE SPACES

In this section we establish some basic technical results for proving that homology groups or homotopy groups of spaces are finitely generated. In order, we discuss spaces with finitely generated homology, groups with finitely generated homology, spaces with finitely generated homotopy groups, and section spaces.

2.1. Homologically finite type spaces. We start with finiteness conditions that one can impose on the homology groups of a space.

Definition 2.1.

- A path-connected space X is said to be of *homologically finite type* if for all $\mathbb{Z}[\pi_1(X)]$ -modules M that are finitely generated as an abelian group, $H_*(X, M)$ is finitely generated in each degree.
- A space X is said to be of *homologically finite type* if it has finitely many path components and each of these path components is of homologically finite type. We use the notation $X \in \text{HFin}$.

Using cellular homology, one sees that a CW-complex with finitely many cells in each dimension has finitely generated homology groups. The following lemma generalizes this:

Lemma 2.2. *A CW complex X with finitely many cells in each dimension is in HFin .*

Proof. A CW decomposition of X induces a $\pi_1(X)$ -equivariant CW decomposition of its universal cover \tilde{X} . Let \tilde{C}_* denote the cellular chains on \tilde{X} . The hypothesis implies this is a finitely generated complex of free $\mathbb{Z}[\pi_1(X)]$ -modules. If M is a $\mathbb{Z}[\pi_1(X)]$ -module, $H_*(X; M)$ is isomorphic to the homology of the chain complex $\tilde{C}_* \otimes_{\mathbb{Z}[\pi_1(X)]} M$. If M is finitely generated as an abelian group, the lemma follows by noting this chain complex has a finitely generated abelian group in each degree. \square

Example 2.3. Any compact n -dimensional topological manifold with boundary is in \mathbf{HFin} . This follows from the fact that such a manifold is a compact metrizable ANR and these have the homotopy type of a finite CW complex [Wes75].

We will next discuss the behavior of \mathbf{HFin} under fiber sequences.

Notation 2.4. A *fiber sequence with fiber taken over $b \in B$* is a pair of maps $F \xrightarrow{i} E \xrightarrow{p} B$ and a null-homotopy from the composite $p \circ i: F \rightarrow B$ to the constant map at b , such that the induced map from $F \rightarrow \mathrm{hofib}_b(E \rightarrow B)$ is a weak equivalence.

Here the homotopy fiber is defined by replacing $E \rightarrow B$ with a fibration $\tilde{p}: \tilde{E} \rightarrow B$ and taking $\mathrm{hofib}_b(E \rightarrow B) = \tilde{p}^{-1}(b)$. This means that the Serre spectral sequence and long exact sequence of homotopy groups apply to fiber sequences. Suppose we are given a fiber sequence $F \rightarrow E \rightarrow B$ with B path-connected and $\pi_1(B)$ acting trivially on $H_*(F)$. Using the Serre spectral sequence one can prove that if two of the three spaces F , E and B have finitely generated homology in each degree, then so does the third one (see for example Lemma 1.9 of [Hat08]). The following lemma generalizes this:

Lemma 2.5. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fiber sequence with B path-connected.*

- (i) *Bases: if $H_*(F)$ is finitely generated in each degree and $H_*(E, p^*M)$ is finitely generated in each degree for all $\mathbb{Z}[\pi_1(B)]$ -modules M that are finitely generated as abelian groups, then $B \in \mathbf{HFin}$.*
- (ii) *Total spaces: if $B \in \mathbf{HFin}$, E is path-connected and $H_*(F, i^*M)$ is finitely generated in each degree for all $\mathbb{Z}[\pi_1(E)]$ -modules M that are finitely generated as abelian groups, then $E \in \mathbf{HFin}$.*
- (iii) *Fibers: if F is simply-connected, $H_*(B)$ and $H_*(E)$ are finitely generated in each degree, and $\pi_1(B)$ acts trivially on $H_*(F)$, then $F \in \mathbf{HFin}$.*

Proof. (i) Suppose we are given a $\mathbb{Z}[\pi_1(B)]$ -module M , then by pulling it back along p we obtain a local coefficient system p^*M on E . Theorem 3.1 of [MS93] describes an equivariant local coefficient Serre spectral sequence. Taking $G = \{e\}$, we apply it to the fiber sequence $F \rightarrow E \rightarrow B$ with local coefficient system p^*M on E and get a spectral sequence

$$E_{p,q}^2 = H_p(B, H_q(F, (p \circ i)^*M)) \Rightarrow H_{p+q}(E, p^*M)$$

Now suppose that M is finitely generated as an abelian group. On the one hand, by hypothesis $H_{p+q}(E, p^*M)$ is finitely generated and thus so are the entries on the E_∞ -page. On the other hand, we claim that $H_q(F, (p \circ i)^*M)$ is finitely generated for all q . To see this, note that the action of $\pi_1(F)$ on $(p \circ i)^*M$ is trivial and hence by the universal coefficient theorem there is a short exact sequence

$$0 \rightarrow H_q(F) \otimes M \rightarrow H_q(F; (p \circ i)^*M) \rightarrow \mathrm{Tor}(H_{q-1}(F), M) \rightarrow 0$$

with both the left and right term finitely generated by our assumptions.

Using these two observations, we prove by induction over n that for all $\mathbb{Z}[\pi_1(B)]$ -modules M that are finitely generated as abelian groups, the groups $H_p(B, M)$ for $p \leq n$ are finitely generated. The initial case is $n = 0$. Then we have that $H_0(B, M)$ is given by the coinvariants $M_{\pi_1(B)}$, and coinvariants of M are finitely generated if M is, since they are given by taking a quotient.

For the induction step we assume the case n and prove the case $n + 1$. Since $H_q(F, (p \circ i)^* M)$ is finitely generated for all q , the entries $E_{p,q}^2 = H_p(B, H_q(F, (p \circ i)^* M))$ on the E^2 -page of the Serre spectral sequence are finitely generated for $p \leq n$ by the inductive hypothesis. The entries on further pages are obtained by taking kernel modulo image, so we have that $E_{p,q}^k$ is finitely generated for $p \leq n$ and $k \geq 2$, i.e. the first n columns are.

We next claim that for all $k \geq 2$, the entry $E_{n+1,0}^k$ on the first row is finitely generated if and only if $E_{n+1,0}^2$ is finitely generated. The proof is by induction over k , the case $k = 2$ being obvious. For the induction step, we assume the case k and prove the case $k + 1$. We have that

$$E_{n+1,0}^{k+1} = \ker(d^k: E_{n+1,0}^k \rightarrow E_{n+1-k,k-1}^k)$$

and since the target is finitely generated, $E_{n+1,0}^{k+1}$ is finitely generated if and only if $E_{n+1,0}^k$ is. By the induction hypothesis the latter happens if and only if $E_{n+1,0}^2$ is finitely generated.

The Serre spectral sequence is first-quadrant, so that $E_{n+1,0}^{n+2} \cong E_{n+1,0}^\infty$. The latter is a subquotient of the finitely generated abelian group $H_{n+1}(E, p^* M)$, so we conclude that $E_{n+1,0}^{n+2}$ is finitely generated. Using the previous claim we conclude that $E_{n+1,0}^2 = H_{n+1}(B, M)$ is finitely generated as well.

- (ii) This is similar to (i) but easier, using the Serre spectral sequence

$$E_{p,q}^2 = H_p(B, H_q(F, i^* M)) \Rightarrow H_{p+q}(E, M)$$

By the hypothesis all entries on the E^2 -page are finitely generated. The entries on further pages of the spectral sequence are obtained by taking kernel modulo image, so $E_{p,q}^k$ is finitely generated for all $k \geq 2$, p and q . Since the Serre spectral sequence is first-quadrant, the entries on the line $p + q$ stabilize after the $(n + 1)$ -page and we conclude that $E_{p,q}^\infty$ is finitely generated for all p and q . Since $H_n(E, M)$ is an iterated extension of the entries $E_{p,q}^\infty$ with $p + q = n$, it is finitely generated as well.

- (iii) This is similar to (i), using the Serre spectral sequence

$$E_{p,q}^2 = H_p(B, H_q(F, A)) \Rightarrow H_{p+q}(E, A)$$

with A a finitely generated abelian group. We will not give a detailed proof, since this case is not used in the paper. \square

2.2. Homologically finite type groups. We now apply the results of the previous subsection to classifying spaces of groups.

Lemma 2.6. *If G is a topological group with underlying space in \mathbf{HFin} , then $BG \in \mathbf{HFin}$.*

Proof. This follows by applying part (i) of Lemma 2.5 to the fiber sequence $G \rightarrow EG \rightarrow BG$. \square

Example 2.7. Since the orthogonal group $O(n)$ is a compact manifold, Lemma's 2.2 and 2.6 imply that $BO(n) \in \mathbf{HFin}$.

The following lemma applies Lemma 2.5 to classifying spaces of discrete groups:

Lemma 2.8. *The class of groups with classifying space in \mathbf{HFin} is closed under the following operations:*

- (i) *Quotients: if $1 \rightarrow H \rightarrow G \rightarrow G' \rightarrow 1$ is a short exact sequence of groups and $BH, BG \in \mathbf{HFin}$, then $BG' \in \mathbf{HFin}$.*
- (ii) *Extensions: if $1 \rightarrow H \rightarrow G \rightarrow G' \rightarrow 1$ is a short exact sequence of groups and $BH, BG' \in \mathbf{HFin}$, then $BG \in \mathbf{HFin}$. This includes products as trivial extensions.*
- (iii) *Finite index subgroups: if $G' \subset G$ has finite index then $BG' \in \mathbf{HFin}$ if and only if $BG \in \mathbf{HFin}$.*

Proof. (i) This follows from part (i) of Lemma 2.5, since a short exact sequence of groups $H \rightarrow G \rightarrow G'$ induces a fiber sequence of classifying spaces $BH \rightarrow BG \rightarrow BG'$.

(ii) Similarly, this follows by applying part (ii) of Lemma 2.5 to $BH \rightarrow BG \rightarrow BG'$.

(iii) The direction \Leftarrow follows from part (ii) of Lemma 2.5 applied to $G/G' \rightarrow BG' \rightarrow BG$.

We first prove the direction \Rightarrow when G' is normal in G . We remark that G/G' is finite, so that by Lemma 2.6 we have that $B(G/G') \in \mathbf{HFin}$. The result then follows from part (ii) of this lemma applied to $1 \rightarrow G' \rightarrow G \rightarrow G/G' \rightarrow 1$.

To reduce the general case to G' normal, note that for any finite index subgroup $G' \subset G$ there is a finite index subgroup $H \subset G'$ such that $H \subset G$ is normal. That $BH \in \mathbf{HFin}$ follows from the direction \Leftarrow . \square

In particular, whether a group has a classifying space in \mathbf{HFin} is independent under changes by finite groups.

Definition 2.9. Two groups G, G' *differ by finite groups*, or equivalently are said to be *virtually isomorphic*, if there is a finite zigzag of homomorphisms with finite kernel and cokernel:

$$G \leftarrow G_1 \rightarrow \dots \leftarrow G_k \rightarrow G'$$

Remark 2.10. A related definition is the following: two groups G, G' are said to be *commensurable* if there is a group H and injective homomorphisms $G \hookrightarrow H \hookrightarrow G'$, each with finite cokernel. Clearly commensurable groups differ by finite groups. Using the fact that an intersection of two finite index subgroups is a finite index subgroup, one can deduce that if G and G' differ by finite groups, they are commensurable.

This is a second class of groups with classifying space in \mathbf{HFin} : arithmetic groups. We use the definition in [Ser79]. In particular, we do *not* assume the \mathbb{Q} -algebraic group G is reductive.

Definition 2.11.

- A subgroup Γ of a \mathbb{Q} -algebraic group $G \subset GL_n(\mathbb{Q})$ is *arithmetic* if the intersection $\Gamma \cap GL_n(\mathbb{Z})$ has finite index in both Γ and $G \cap GL_n(\mathbb{Z})$.
- A group Γ is *arithmetic* if it can be embedded in a \mathbb{Q} -algebraic group G as an arithmetic subgroup.

Examples of arithmetic groups include all finite groups, all finitely generated abelian groups and all torsion-free finitely generated nilpotent groups, see Section 1.2 of [Ser79].

Theorem 2.12 (Borel-Serre). *If G is an arithmetic group, then $BG \in \mathbf{HFin}$.*

Proof. By property (5) of Section 1.3 of [Ser79], every torsion-free arithmetic group Γ has a $B\Gamma$ that is a finite CW complex and using Lemma 2.2 we conclude such groups have a classifying space in \mathbf{HFin} . By property (4) of Section 1.3 of [Ser79] every arithmetic group G has a finite index torsion-free subgroup Γ , so from part (iii) of Lemma 2.8 it follows that any arithmetic group G has $BG \in \mathbf{HFin}$. \square

2.3. Homotopically finite type spaces. Alternatively we can impose finiteness conditions on its homotopy groups. The following more subtle distinction will be useful.

Definition 2.13. • We say X is of *homotopically finite type* if it has finitely many path components and each of its path components has the property that $B\pi_1 \in \mathbf{HFin}$ and π_i is finitely generated for $i \geq 2$. We use the notation $X \in \Pi\mathbf{Fin}$.
• We say X is of *finite type* if it has finitely many path components and each of its path components has the property that π_1 is finite and π_i is finitely generated for $i \geq 2$. We use the notation $X \in \mathbf{Fin}$.

By Lemma 2.6, finite groups have classifying spaces in \mathbf{HFin} , so that $\mathbf{Fin} \subset \Pi\mathbf{Fin}$. In the following lemma's we prove that $\Pi\mathbf{Fin} \subset \mathbf{HFin}$ and that $\mathbf{HFin} \cap \{\pi_1 \text{ is finite}\} \subset \mathbf{Fin}$. This uses the Postnikov tower of a space, see Section VI.2 of [GJ09]. Let us assume X is a path-connected space, then the n th stage of the Postnikov tower is $P_n(X) = \|\cosk_n \text{Sing}(X)\|$. We use the following two properties of this construction. Firstly, the homotopy groups of $P_n(X)$ are given by

$$\pi_i(P_n(X)) = \begin{cases} \pi_i(X) & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases}$$

and secondly there are fiber sequences

$$(3) \quad K(\pi_n(X), n) \rightarrow P_n(X) \rightarrow P_{n-1}(X)$$

Lemma 2.14. *If A is a finitely generated abelian group and $n \geq 1$, then $K(A, n) \in \mathbf{HFin}$.*

Proof. The proof is by induction over n . In the initial case $n = 1$, we have that $K(A, 1) = BA \in \mathbf{HFin}$ by Theorem 2.12, because finitely generated abelian groups are arithmetic. For the induction step, apply part (i) of Lemma 2.5 to the fiber sequence $K(A, n-1) \rightarrow * \rightarrow K(A, n)$. \square

Lemma 2.15. *If $X \in \Pi\mathbf{Fin}$, then $X \in \mathbf{HFin}$.*

Proof. Without loss of generality X is path-connected. We have that $P_0(X) \simeq *$, so that $P_1(X) \simeq K(\pi_1(X), 1) \simeq B\pi_1(X) \in \mathbf{HFin}$ by assumption. To finish the proof, apply Lemma 2.14 and part (ii) of Lemma 2.5 inductively to the fiber sequence (3). \square

We continue with the proof that $\mathbf{HFin} \cap \{\pi_1 \text{ is finite}\} \subset \mathbf{Fin}$.

Lemma 2.16. *If $X \in \mathbf{HFin}$ and each component is simply-connected, then $X \in \mathbf{Fin}$.*

Proof. Without loss of generality X is path-connected. Our proof is by induction over n of the statement that π_i with $i \leq n$ is finitely generated. The initial case $n = 1$ follows from the fact that X is simply-connected. Suppose we have proven the case n , then we will prove the case $n + 1$.

We claim that $H_*(P_{n+1}(X))$ is finitely generated for $* \leq n+2$. We start with $* = n+1, n+2$: the map $X \rightarrow P_{n+1}(X)$ is $(n+2)$ -connected (i.e. an isomorphism on π_i for $i \leq n+1$ and a surjection for $* = n+2$), so that in particular $H_*(X)$ surjects onto $H_*(P_{n+1}(X))$ for $* = n+1, n+2$ and thus the latter are finitely generated. For $* \leq n$, note that $P_{n+1}(X) \rightarrow P_n(X)$ is $(n+1)$ -connected so that in particular $H_*(P_{n+1}(X)) \cong H_*(P_n(X))$ for $* \leq n$. By the inductive hypothesis $P_n(X)$ is simply-connected with finitely generated higher homotopy groups, so in \mathbf{HFin} by Lemma 2.15. The long exact sequence for homology of a pair then tells us that $H_{n+2}(P_n(X), P_{n+1}(X))$ is also finitely generated. The relative Hurewicz theorem says that

$$H_{n+2}(P_n(X), P_{n+1}(X)) \cong \pi_{n+2}(P_n(X), P_{n+1}(X)) \cong \pi_{n+1}(X)$$

which completes the proof. \square

Lemma 2.17. *If $X \in \mathbf{HFin}$ and π_1 of each component is finite, then $X \in \mathbf{Fin}$.*

Proof. Without loss of generality X is path-connected. Let \tilde{X} denote the universal cover of X , then we start by proving $\tilde{X} \in \mathbf{HFin}$. We have $H_*(\tilde{X}; M) \cong H_*(X, \mathbb{Z}[\pi_1(X)] \otimes M)$, so $\tilde{X} \in \mathbf{HFin}$ follows directly from the assumptions that $X \in \mathbf{HFin}$ and that $\pi_1(X)$ is finite. The higher homotopy groups of \tilde{X} are those of X , so it suffices to show that $\pi_i(\tilde{X})$ is finitely generated. This follows from Lemma 2.16. \square

If X is path-connected, let $X\langle n \rangle$ denote the n -connective cover, the homotopy fiber of $X \rightarrow P_n(X)$. This is well-defined up to homotopy. For example, $X\langle 1 \rangle$ is the universal cover \tilde{X} . The n -connective cover has the following property:

$$\pi_i(X\langle n \rangle) = \begin{cases} 0 & \text{if } i \leq n \\ \pi_i(X) & \text{if } i > n \end{cases}$$

Lemma 2.18. *If X is path-connected, $\pi_1(X)$ is finite and $X \in \mathbf{HFin}$, then $X\langle n \rangle \in \mathbf{Fin}$ for all $n \geq 0$.*

Proof. By Lemma 2.17 we can replace \mathbf{HFin} by \mathbf{Fin} and in particular the homotopy groups of X are finitely generated. Since $\pi_i(X\langle n \rangle)$ is given by 0 for $i \leq n$ and $\pi_i(X)$ for $i > n$, it is finitely generated. \square

Example 2.19. Since $\pi_0(O(n)) \cong \mathbb{Z}/2\mathbb{Z}$, from Example 2.7 and Lemma 2.17 we conclude that $BO(n) \in \mathbf{Fin}$. Using Lemma 2.18 we conclude that $BO(2n)\langle n \rangle \in \mathbf{Fin}$ as well.

One convenience of working with $\mathbf{\Pi Fin}$ or \mathbf{Fin} is that we can use the long exact sequence of homotopy groups, the precise statement of which is as follows. Let $p: E \rightarrow B$ be a Serre fibration, pick a base point $e_0 \in E$ and let F be the fiber of p over $p(e_0)$. Then there is a long exact sequence

$$(4) \quad \cdots \rightarrow \pi_1(F, e_0) \rightarrow \pi_1(E, e_0) \rightarrow \pi_1(B, p(e_0)) \rightarrow \pi_0(F) \rightarrow \pi_0(E) \rightarrow \pi_0(B)$$

where the last three entries are pointed sets, with path components containing e_0 and $p(e_0)$ providing the base points.

Lemma 2.20. *Let $p: E \rightarrow B$ be a Serre fibration, $e_0 \in E$ a base point and $F = p^{-1}(p(e_0))$. Let F_0 , E_0 and B_0 denote the path components of F , E and B containing e_0 , e_0 and $p(e_0)$ respectively.*

Then the following holds for π_i for $i \geq 2$:

- (i) Bases: if F_0 has finitely generated π_i for $i \geq 2$ and E_0 has finitely generated π_i for $i \geq 3$, then B_0 has finitely generated π_i for $i \geq 3$. If additionally $\pi_2(E_0)$ is finitely generated and either
 - (a) $\pi_1(F_0)$ is finite,
 - (b) $\pi_1(F_0)$ is a finitely generated nilpotent group, or
 - (c) $\pi_1(F_0)$ is finitely generated and $\pi_1(E_0)$ is finite,
 then $\pi_2(B_0)$ is finitely generated.
- (ii) Total spaces: if F_0, B_0 have finitely generated π_i for $i \geq 2$ then so does E_0 .
- (iii) Fibers: if E_0 has finitely generated π_i for $i \geq 2$ and B_0 has finitely generated π_i for $i \geq 3$, then F_0 has finitely generated π_i for $i \geq 2$.

Furthermore, the following holds for π_1 :

- (i') Bases: if $\pi_1(E_0)$ is finitely generated and $\pi_0(F)$ is finite, then $\pi_1(B_0)$ is finitely generated.
- (ii') Total spaces: if $\pi_1(F_0)$ and $\pi_1(B_0)$ are finitely generated and either
 - (a) $\pi_1(B_0)$ is finite,
 - (b) $\pi_1(B_0)$ is nilpotent, or
 - (c) $\pi_0(F)$ is finite,
 then $\pi_1(E_0)$ is finitely generated.
- (iii') Fibers: if $\pi_2(B_0)$ and $\pi_1(E_0)$ are finitely generated and either
 - (a) $\pi_1(E_0)$ is finite,
 - (b) $\pi_1(E_0)$ is nilpotent, or
 - (c) $\pi_1(B_0)$ is finite,
 then $\pi_1(F_0)$ is finitely generated.

Proof. Since π_i for $i \geq 1$ depends only the component containing the base point, the long exact sequence (4) is given by

$$\cdots \rightarrow \pi_2(B_0, p(e_0)) \rightarrow \pi_1(F_0, e_0) \rightarrow \pi_1(E_0, e_0) \rightarrow \pi_1(B_0, p(e_0)) \rightarrow \pi_0(F)$$

Parts (ii), (iii) and the first claim in (i) are a consequence of this long exact sequence and the fact that in the category of abelian groups, the class of finitely generated abelian groups is closed under taking subgroups, quotients and extensions. This argument fails for the second claim in part (i), i.e. for $\pi_2(B_0)$, because it only says that $\pi_2(B_0)$ is an extension of a subgroup of $\pi_1(F_0)$ by a quotient of $\pi_2(E_0)$, but subgroups of finitely generated non-abelian groups need not be finitely generated. The second part of (i) gives conditions to guarantee this problem does not occur, using the following facts:

- (a) a subgroup of a finite group is finite,
- (b) a subgroup of a finitely generated nilpotent group is a finitely generated nilpotent group,
- (c) if a group is finitely generated if and only if a finite index subgroup is finitely generated.

Part (i'), (ii') and (iii') are obtained by applying these same facts and using that finitely generated groups are closed under extensions. \square

2.4. Section spaces. We continue with examples of spaces that can be proven to be in Fin or ΠFin with relative ease. If $E \rightarrow B$ is a Serre fibration, we let $\Gamma(E, B)$ denote the space

of sections in the compact-open topology. Given a section s , we have a relative version: if $A \subset B$ is a subspace, then $\Gamma(E, B; A) \subset \Gamma(E, B)$ is the subspace of sections that are equal to s on A .

Lemma 2.21. *Let $E \rightarrow B$ be a Serre fibration with fiber F and section $s: B \rightarrow E$. Suppose that B is a path-connected finite CW-complex, $A \subset B$ is a non-empty subcomplex of B , and each component of F has finitely generated homotopy groups. Then each component of $\Gamma(E, B; A)$ has finitely generated homotopy groups.*

The condition that A is non-empty can be dropped if π_1 of each component of F is finite or nilpotent, or F has finitely many components.

Proof. We prove the first part of the lemma by induction over the number k of cells in B that are not in A . In the initial case $k = 0$, $\Gamma(E, B; B) = \{s\}$. For the induction step, suppose we have proven the case k , then we will prove the case $k + 1$. That is, we have $A' = A \cup_{S^{d-1}} D^d$ where there are k cells of B not in A' . The induction hypothesis thus says the claim is true for A' , and we want to prove it for A .

There are two cases, the first of which is $d \geq 1$. Restriction of sections to D^d fits into a fiber sequence

$$\Gamma(E, B; A \cup D^d) \rightarrow \Gamma(E, B; A) \rightarrow \Gamma(E|_{D^d}, D^d; S^{d-1})$$

with fiber taken over $s|_{D^d} \in \Gamma(E|_{D^d}, D^d; S^{d-1})$.

We apply the induction hypothesis to the fiber, which says each component of $\Gamma(E, B; A \cup D^d)$ has finitely generated π_i for $i \geq 1$. The base $\Gamma(E|_{D^d}, D^d; S^{d-1})$ is equivalent to $\Omega^d F$, whose components have finitely generated π_i for $i \geq 1$ by our hypothesis. The result now follows from the parts (ii) and (ii') of Lemma 2.20. For part (ii') we use that condition (b) is satisfied: $\pi_1(\Omega^d F)$ is abelian if $d \geq 1$.

The second case is $d = 0$. Then we have that $A' = A \sqcup D^0$. Since B is path-connected, by adding some 1-cells to A' we can obtain a subcomplex A'' of B containing A' such that the inclusion $A \hookrightarrow A''$ is a homotopy equivalence. This implies that the inclusion map $\Gamma(E, B; A'') \rightarrow \Gamma(E, B; A)$ is a weak equivalence, which completes the proof since A'' has more cells than A and hence we can apply the inductive hypothesis.

For the second claim, we consider the fiber sequence

$$\Gamma(E, B; D^0) \rightarrow \Gamma(E, B) \rightarrow E|_{D^0}$$

with fiber taken over $s \in E|_{D^0}$. By the first part, each component of $\Gamma(E, B; D^0)$ has finitely generated homotopy groups. Furthermore, the base is equivalent to F . Thus the result follows part (ii) and (ii') of Lemma 2.20, using that condition (a), (b) or (c) holds by assumption. \square

3. SELF-EMBEDDINGS

In this section we study spaces of self-embeddings. Let M be a smooth n -dimensional connected manifold with non-empty boundary ∂M and an embedded disk $D^{n-1} \subset \partial M$. Let $\text{Emb}_{1/2\partial}(M)$ be the space of embeddings $M \hookrightarrow M$ that are the identity on $\partial M \setminus \text{int}(D^{n-1})$. They do *not* need to take the entire boundary to itself. There is an inclusion $\text{Diff}_\partial(M) \rightarrow \text{Emb}_{1/2\partial}(M)$ and its image is contained in the subspace of embeddings isotopic to a diffeomorphism that is the identity on the boundary:

Definition 3.1. We let $\text{Emb}_{1/2\partial}^{\cong}(M)$ denote the subspace of $\text{Emb}_{1/2\partial}(M)$ consisting of those embeddings e that are isotopic through embeddings e_t fixing $\partial M \setminus \text{int}(D^{n-1})$ to a diffeomorphism e_1 of M which fixes ∂M .

That is, $\text{Emb}_{1/2\partial}^{\cong}(M)$ is the union of the path components of $\text{Emb}_{1/2\partial}(M)$ in the image of $\text{Diff}_{\partial}(M)$. Composition gives $\text{Emb}_{1/2\partial}^{\cong}(M)$ the structure of a topological monoid, and hence its connected components are a monoid. However, since every path component is represented by a diffeomorphism and hence has an inverse, $\pi_0(\text{Emb}_{1/2\partial}^{\cong}(M))$ is in fact a group. The main result of this section is the following:

Theorem 3.2. *Suppose that $n \geq 6$, M is 2-connected and $\partial M = S^{n-1}$. Then we have that $B\text{Emb}_{1/2\partial}^{\cong}(M) \in \Pi\text{Fin}$ and hence HFin .*

Proof. In Proposition 3.3 we prove that the identity component is in ΠFin and in Proposition 3.8 we prove that the classifying space of the group of components is in HFin . By definition we get $B\text{Emb}_{1/2\partial}^{\cong}(M) \in \Pi\text{Fin}$. \square

3.1. The group of path components. We start by studying the group of path components using results of Sullivan.

Proposition 3.3. *Let $n \geq 5$ and suppose we are given an n -dimensional oriented smooth manifold M with finite fundamental group and $\partial M = S^{n-1}$. Then $\pi_0(\text{Diff}_{\partial}(M))$ and $\pi_0(\text{Emb}_{1/2\partial}^{\cong}(M))$ have classifying spaces in HFin .*

Proof. We prove that $\pi_0(\text{Diff}_{\partial}(M))$ is an arithmetic group, so Theorem 2.12 implies the first part. Let $N = M \cup_{S^{n-1}} D^n$ and $\text{Emb}^+(D^n, M)$ denote the orientation-preserving embeddings. Then there is a fiber sequence

$$\text{Diff}_{\partial}(M) \rightarrow \text{Diff}^+(N) \rightarrow \text{Emb}^+(D^n, N)$$

with fiber taken over the standard embedding $D^n \hookrightarrow N$. The space $\text{Emb}^+(D^n, N)$ has one component, and by hypothesis on $\pi_1(N)$ has finite π_1 . Thus $\pi_0(\text{Diff}_{\partial}(M))$ differs by finite groups from $\pi_0(\text{Diff}^+(N))$. Sullivan proved that for closed oriented 1-connected N of dimension ≥ 5 , $\pi_0(\text{Diff}^+(N))$ is commensurable with an arithmetic group [Sul77, Theorem 13.3] and Triantafyllou generalized this to oriented manifolds with finite fundamental groups [Tri95].

For $\pi_0(\text{Emb}_{1/2\partial}^{\cong}(M))$, note that Theorem 4.20 implies there is an exact sequence of groups

$$\pi_0(\text{Diff}_{\partial}(D^n)) \rightarrow \pi_0(\text{Diff}_{\partial}(M)) \rightarrow \pi_0(\text{Emb}_{1/2\partial}^{\cong}(M)) \rightarrow 1$$

Since $\pi_0(\text{Diff}_{\partial}(D^n)) \cong \Theta_{n+1}$ is finite, $\pi_0(\text{Diff}_{\partial}(M))$ and $\pi_0(\text{Emb}_{1/2\partial}^{\cong}(M))$ differ by finite groups and first part suffices. \square

3.2. A recollection of embedding calculus. Embedding calculus is the “pointillistic” study of embeddings [Wei15, Remark 4.5.4]. That is, think of an embedding as an immersion that has a certain property when evaluated on finite subsets of its domain: it is injective. If we replace this condition with homotopy-theoretic data, we get an object accessible to homotopy theory. Multiple disjunction theory for manifolds as in [GK15] says that the space of these homotopy-theoretic alternatives is the same as the space of embeddings as long as the codimension is at least 3.

We will use two types of embedding calculus: the *sheafification* approach of [Wei99] and the *configuration category* approach of [BdBW15]. For our purposes the former has a calculational advantage, while the latter has a formal advantage. For convenience we will leave out boundary conditions, as the modifications to the statements are straightforward (though the proofs may not be), see Section 10 of [Wei99] and Section 6 of [BdBW15].

3.2.1. *The sheafification approach.* Fix two manifolds M and N . The embedding calculus of [Wei99] gives a Taylor tower

$$\begin{array}{ccc}
 & & \vdots \\
 & \nearrow & \downarrow \\
 \text{Emb}(M, N) & \longrightarrow & T_k(\text{Emb}(M, N)) \\
 & \searrow & \downarrow \\
 & & T_{k-1}(\text{Emb}(M, N)) \\
 & \searrow & \downarrow \\
 & & \vdots
 \end{array}$$

starting at $T_1(\text{Emb}(M, N))$. The k th Taylor approximation $\text{Emb}(M, N) \rightarrow T_k(\text{Emb}(M, N))$ is the best approximation to $\text{Emb}(M, N)$ built out of the restrictions of embeddings to $\leq k$ disks in M . The precise description of this on page 84 of [Wei99] makes clear the existence of the tower. This tower has the following properties:

- Suppose M has handle dimension h , then $\text{Emb}(M, N) \rightarrow T_k(\text{Emb}(M, N))$ is $-(n-1) + k(n-2-h)$ -connected. This is Corollary 2.6 of [GW99], but depends on the results of [GK15].
- $T_1(\text{Emb}(M, N))$ is the space $\text{Imm}(M, N)$ of immersions. This appears on page 97 of [Wei99].
- Fix an embedding $\iota: M \hookrightarrow N$. For a finite set I , let $F_I(M) = \text{Emb}(I, M)$ be the ordered configuration space. Let $C_k(M) = F_{\{1, \dots, k\}}(M)/\mathfrak{S}_k$ be the unordered configuration space.

For $k \geq 2$, the homotopy fiber of $T_k(\text{Emb}(M, N)) \rightarrow T_{k-1}(\text{Emb}(M, N))$ over the image ι is weakly equivalent to a relative section space: the space of sections of the bundle over $C_k(M)$ with fiber over a configuration $c \in C_k(M)$ given by $\text{tohofib}_{I \subset c}(F_I(N))$ that are equal to a section s^t near the fat diagonal. We describe the section s^t by giving a base point in each $F_I(N)$: recall that I is a collection of points in M , then it is the inclusion of I into N by ι . This description appears as Theorem 9.2 of [Wei99].

The last point used total homotopy fibers, whose definition we recall. Let $[1]$ be the category $0 \rightarrow 1$, so that a functor $T: [1]^k \rightarrow \mathbf{Top}_*$ is a k -dimensional cube of pointed spaces. The total homotopy fiber of T is given by

$$\text{tohofib}(T) = \text{hofib}[T(0, \dots, 0) \rightarrow \text{holim}_{[1]^k \setminus (0, \dots, 0)} T]$$

It can also be computed by iteratively taking homotopy fibers in each of the k directions. In the non-pointed setting, if $T: [1]^k \rightarrow \mathbf{Top}$ has $T(0, \dots, 0)$ path-connected, then $\text{tohofib}(T)$ is well-defined up to homotopy.

3.2.2. The configuration category approach. In [BdBW15], Weiss and Boavida de Brito reformulated embedding calculus in a more categorical manner. The categories of interest are so-called configuration categories $\mathbf{con}(M)$, which are ∞ -categories modeled by Segal spaces over the nerve of the category of finite sets.

There are various models for $\mathbf{con}(M)$, one of which is the particle model. Roughly speaking, this has objects given by ordered configurations of a finite set in M , and morphisms obtained from forgetting particles and reverse “sticky paths of configurations”, i.e. points may start together and then split up never to collide again. The map to the nerve of finite sets remembers the finite set. An embedding $M \hookrightarrow N$ induces a functor $\mathbf{con}(M) \rightarrow \mathbf{con}(N)$ by the induced map on configurations and paths of sticky configurations. By taking inverse image of the category of finite sets of cardinality $\leq k$, we obtain a filtration $\mathbf{con}(M; k)$ of $\mathbf{con}(M)$ by number of particles.

There is also a local version $\mathbf{con}^{\text{loc}}(M)$ of the configuration category with similar filtration $\mathbf{con}^{\text{loc}}(M; k)$. By Lemma 4.2 of [BdBW15] this local version is obtained by applying a functor $L(-)$ to Segal spaces over $N(\mathbf{Fin})$. Theorem 2.1.1 of [Wei15] then says there exists maps such that the following diagram is homotopy cartesian for $k \geq 1$:

$$(5) \quad \begin{array}{ccc} T_k(\text{Emb}(M, N)) & \longrightarrow & \mathbb{R}\text{Map}_{N(\mathbf{Fin})}(\mathbf{con}(M; k), \mathbf{con}(N; k)) \\ \downarrow & & \downarrow \\ \text{Imm}(M, N) & \longrightarrow & \mathbb{R}\text{Map}_{N(\mathbf{Fin})}(\mathbf{con}^{\text{loc}}(M; k), \mathbf{con}^{\text{loc}}(N; k)) \end{array}$$

Here $\mathbb{R}\text{Map}$ denotes the derived mapping space. One way to define this derived mapping space is to pick the simplicial model category structure on Segal spaces over the nerve of the category of finite sets as defined in Appendix B of [BdBW15], cofibrantly and fibrantly replace both the domain and the target, and take the space of maps between these. In particular, derived endomorphism spaces can be modeled by topological monoids. Using this remark, the categorical approach allows us to prove the following formal property of the Taylor approximations.

Lemma 3.4. *The identity component of $T_k(\text{Emb}_{1/2\partial}(M))$ has nilpotent π_1 .*

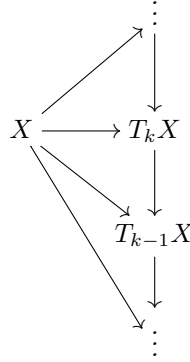
Proof. Remark that in the diagram 5, all corners except $T_k(\text{Emb}_{1/2\partial}(M))$ can be modeled by topological monoids. The result then follows by noting that path-connected H -spaces have abelian π_1 . \square

Remark 3.5. In fact, we believe that $T_k(\text{Emb}_{1/2\partial}(M))$ is an \mathbf{A}_∞ -space and hence has abelian π_1 . We believe this might be helpful for further study of spaces of self-embeddings; it implies that rationally the identity path components of the Taylor approximations are formal and have free graded-commutative algebras as rational (co)homology.

3.3. Identity component. The identity component will be studied using the sheafification approach embedding calculus. The following lemma explains how to use towers of approximations to prove that a space is in ΠFin . Note condition (iii) requires certain path components to have nilpotent π_1 , the reason for proving Lemma 3.4.

Lemma 3.6. *Suppose that a path-connected space X with base point x has the following properties:*

(i) *There is a tower*



starting at $T_0 X$, such that the map from X to the path component of $T_k X$ containing the image of x is $f(k)$ -connected with $\limsup_{k \rightarrow \infty} f(k) = \infty$.

(ii) *The path component of $T_0 X$ containing the image of x has finitely generated homotopy groups.*

(iii) *The path component of $T_k X$ containing the image of x has finite or nilpotent π_1 .*

(iv) *The homotopy fiber of $T_k X \rightarrow T_{k-1} X$ over the image of x is weakly equivalent to a relative section space $\Gamma_k = \Gamma(E_k, B_k; A_k)$ of a fibration $E_k \rightarrow B_k$ with a section, where B_k is a path-connected finite CW-complex, $A_k \subset B_k$ is a non-empty subcomplex and each component of F_k has finitely generated homotopy groups.*

Then $X \in \Pi\text{Fin}$.

Proof. Conditions (i) and (iii) imply that $\pi_1(X)$ will be finite or nilpotent. Theorem 2.12 implies finite groups and finitely generated nilpotent groups have classifying spaces in HFin , so it suffices to prove that $\pi_i(X)$ is finitely generated for $i \geq 1$.

Let $x_k \in T_k X$ denote the image of X . Using hypothesis (i), if we care about a fixed homotopy group $\pi_i(X)$, we can assume that the tower ends at some finite stage $T_K X$ and do an induction over K of the statement that the component of $T_K X$ containing x_K has finitely generated homotopy groups. The initial case $K = 0$ is provided by hypothesis (ii).

For the induction step we assume that case k and prove the case $k + 1$, we use the long exact sequence of homotopy groups from the fiber sequence in hypothesis (iv):

$$\cdots \rightarrow \pi_i(\Gamma_{k+1}, x_{k+1}) \rightarrow \pi_i(T_{k+1} X, x_{k+1}) \rightarrow \pi_i(T_k X, x_k) \rightarrow \cdots$$

where without loss of generality we may assume $x_{k+1} \in T_{k+1} X$ is the image of $x_{k+1} \in \Gamma_{k+1}$. Using hypothesis (iii), Lemma 2.21 says each component of Γ_{k+1} has finitely generated homotopy groups. The induction hypothesis says that the path component of $T_k X$ containing x_k has finitely generated homotopy groups. We then use parts (ii) and (ii') of Lemma 2.20

to finish the proof of the induction step. For part (ii') we use that condition (a) or (b) holds by hypothesis (iii). \square

Embedding calculus has the requirement that certain bounds on the dimensions of the handles in a handle decomposition of M are satisfied. The following lemma gives a situation when that is the case.

Lemma 3.7. *Suppose that $n \geq 6$, M is 2-connected and $\partial M = S^{n-1}$. Then M has a handle decomposition relative to $\partial M \setminus \text{int}(D^{n-1})$ with only handles of dimension $< n - 2$.*

Proof. Pick a disk $D^n \subset M$ and write $W = M \setminus \text{int}(D^n)$, $V = \partial D^n \cong S^{n-1}$, $V' = \partial M \cong S^{n-1}$. We will prove that W admits a Morse function with value 0 on V and value 1 on V' with no critical points of index 0, 1, 2, $n - 2$, $n - 1$ or n .

Pick a Morse function f on W with value 0 on V and value 1 on V' . Removal of critical points of f with index 0 and 1 is done by Theorem 8.1 of [Mil65]. To next remove the critical points of index 2, we apply the proof of Theorem 7.8 of [Mil65], which requires $H_2(W, V) = 0$. But this homology group is the same as $H_2(M, D^n) = H_2(M) = 0$. A similar argument with V' works to remove the critical points of index $n - 2$, $n - 1$ or n . \square

Proposition 3.8. *Suppose that $n \geq 5$ and M has a handle decomposition rel $\partial M \setminus \text{int}(D^{n-1})$ with handles of dimension $h < n - 2$. Then the identity component $\text{Emb}_{1/2\partial}^{\text{id}}(M)$ of $\text{Emb}_{1/2\partial}(M)$ is in ΠFin .*

Proof. We will apply Lemma 3.6. The tower comes from the embedding calculus tower in the sheafification approach, extended once at the bottom by Smale-Hirsh.¹

$$\begin{array}{ccc}
 & & \vdots \\
 & \nearrow & \downarrow \\
 \text{Emb}_{1/2\partial}^{\text{id}}(M) & \longrightarrow & T_k(\text{Emb}_{1/2\partial}(M)) \\
 & \searrow & \downarrow \\
 & & T_{k-1}(\text{Emb}_{1/2\partial}(M)) \\
 & & \downarrow \\
 & & \vdots
 \end{array}$$

We recall its description from Subsection 3.2.1 and check the conditions in Lemma 3.6 hold:

- (i) If M has handle dimension h , the map from $\text{Emb}_{1/2\partial}^{\text{id}}(M)$ to the path component of $T_k(\text{Emb}_{1/2\partial}(M))$ containing the image of the identity is $(-(n - 1) + k(n - 2 - h))$ -connected. Since $h < n - 2$, this goes to ∞ as $k \rightarrow \infty$.
- (ii) The identity path component of $T_0(\text{Emb}_{1/2\partial}(M))$ is the space $\text{Map}_{1/2\partial}^{\text{id}}(M)$ of maps $M \rightarrow M$ that are the identity on $\partial M \setminus \text{int}(D^{n-1})$ and homotopic to the identity. Since

¹The reader familiar with embedding calculus may note that our T_0 is not the 0th Taylor approximation. However, we find our notation convenient for this particular argument.

- a mapping space is an example of a section space and simply-connected compact manifolds are in Fin , by Lemma 2.21 $\text{Map}_{1/2\partial}^{\text{id}}(M)$ has π_i for $i \geq 1$ finitely generated.
- (iii) The identity path component of $T_k(\text{Emb}_{1/2\partial}(M))$ has nilpotent π_1 by Lemma 3.4.
 - (iv) The cases $k = 1$ and $k \geq 2$ are different. We start with the former, and have that $T_1(\text{Emb}_{1/2\partial}(M))$ is the space $\text{Imm}_{1/2\partial}(M)$ of immersions $M \looparrowright M$ that are the identity on $\partial M \setminus \text{int}(D^{n-1})$. The map $T_1(\text{Emb}_{1/2\partial}(M)) \rightarrow T_0(\text{Emb}_{1/2\partial}(M))$ is inclusion of immersions into continuous maps. Smale-Hirsh [Sma59a] says that there is a fiber sequence with fiber taken over the identity

$$\Gamma(\text{Iso}(TM), M; \partial M \setminus \text{int}(D^{n-1})) \rightarrow \text{Imm}_{1/2\partial}(M) \rightarrow \text{Map}_{1/2\partial}(M)$$

The fiber given by the space of sections of the bundle over M with fiber over $m \in M$ given by $\text{Iso}(T_m M) \simeq O(2n)$. By Lemma 2.21 the components of the section space $\Gamma(\text{Iso}(TM), M; \partial M \setminus \text{int}(D^{n-1}))$ have finitely generated homotopy groups.

Next we discuss the case $k \geq 2$. Recall that for a finite set I , $F_I(M) = \text{Emb}(I, M)$ is the ordered configuration space. There is also an unordered configuration space $C_k(M) = F_{\{1, \dots, k\}}(M)/\mathfrak{S}_k$. For $k \geq 2$, there is a bundle over $C_k(M)$ with fiber over a configuration $c \in C_k(M)$ given by $\text{tohofib}_{I \subset c}(F_I(M))$. It has a section s^{id} , which can be described by giving compatible base points in the spaces $F_I(M)$; these are given by $\text{id} \in \text{Emb}(I, M) = F_I(M)$, after recalling that I is a collection of points in M . Then the homotopy fiber of $T_k(\text{Emb}_{1/2\partial}(M)) \rightarrow T_{k-1}(\text{Emb}_{1/2\partial}(M))$ is the space of sections of this bundle that are equal s^{id} near the fat diagonal and $\partial M \setminus \text{int}(D^{n-1})$.

First of all, we can replace $C_k(M)$ with its homotopy equivalent Fulton-MacPherson compactification $C_k[M]$ [Sin04]. This is a finite CW complex and the requirement that the sections are equal to s^{id} near both the fat diagonal and $\partial M \setminus \text{int}(D^{n-1})$, then becomes that the sections are equal to s^{id} on a non-empty subcomplex.

Next we prove that the fiber $\text{tohofib}_{I \subset [k]}(F_I(M))$ is in Fin . The Fulton-MacPherson compactification $F_I[M]$ of $F_I(M)$ is a finite CW complex and 1-connected, so by Lemma's 2.2 and 2.17 it is in Fin . By Theorem B of [GK15] $\text{tohofib}_{I \subset [k]}(F_I(M))$ is $-(n-3) + k(n-2)$ -connected, so 1-connected as $k \geq 2$. Since a total homotopy fiber is obtained by iterated homotopy fibers and we can disregard the lower homotopy groups, part (iii) of Lemma 2.20 suffices. \square

4. THE WEISS FIBER SEQUENCE

The goal of this section is to construct the fiber sequence (1) and its delooping (2), both of which were discussed in the introduction. There the delooping was informally described as

$$B\text{Diff}_{\partial}(M) \rightarrow B\text{Emb}_{1/2\partial}^{\cong}(M) \rightarrow B(B\text{Diff}_{\partial}(D^n), \mathfrak{h})$$

It will arise in this section as the homotopy quotient of an action of a topological monoid on a module:

$$\text{BM} \rightarrow \text{BM} // \text{BD} \rightarrow * // \text{BD}$$

Subsection 4.1 discusses the relevant background material on topological monoids and Subsection 4.2 defines BD and BM . In Subsection 4.3 we then construct the fiber sequence and identify its terms. In Subsection 4.4 we give several useful generalizations.

Remark 4.1. There is a choice whether $\text{Diff}_\partial(-)$ consists diffeomorphisms that (a) are the identity on the boundary, (b) are the identity on the boundary and furthermore have all derivatives equal to those of the identity on the boundary, or (c) are the identity on a neighborhood of the boundary. We have $(c) \subset (b) \subset (a)$ and all inclusions are weak equivalences, so we will not distinguish between these.

4.1. Classifying spaces of topological monoids. Though we will recall notation below, we assume that the reader is familiar with topological monoids, simplicial spaces, geometric realization, and the results of Appendix A of [Seg74]. We also assume the reader is familiar with the double bar construction, as in Section 9 of [May72]. Another helpful reference is Section 2 of [Seg78].

A *topological monoid* is a unital monoid object in the category of compactly generated weakly Hausdorff spaces,² i.e. a space A with maps $m: A \times A \rightarrow A$ and $u: * \rightarrow A$ satisfying associativity and unit axioms. A left module B over a topological monoid A is a left module object over A in the same category, i.e. a space B with a map $m: A \times B \rightarrow B$ satisfying associativity and unit axioms. There is a similar definition for right modules.

If A is a topological monoid, B is a left A -module and C is a right A -module, the bar construction is the simplicial space $B_\bullet(B, A, C)$ with p -simplices given by

$$B_p(B, A, C) = B \times A^p \times C$$

The face maps are induced by the monoid multiplication and action maps, and the degeneracy maps induced by the unit of A . Note that $*$ is always a left and right A -module, it is in fact both the terminal left A -module and the terminal right A -module.

Definition 4.2. Let A be a topological monoid and B be a right A -module. Then the *homotopy quotient* $B // A$ is defined to be the geometric realization of the simplicial space given by the double bar construction $B_\bullet(B, A, *)$.

For constructions using topological monoids to be well-behaved, we need to impose some conditions on their unit elements. In this paper, a cofibration means a closed Hurewicz cofibration. A topological monoid A is *well-pointed* if the inclusion $\{e\} \rightarrow A$ of the unit is a cofibration. A simplicial space X_\bullet is *good* if each map $s_i(X_q) \rightarrow X_{q+1}$ is a cofibration and it is *proper* if the map $\bigcup_i s_i(X_q) \rightarrow X_{q+1}$ is a cofibration. Good implies proper. A basic but extremely useful property of proper simplicial spaces is that if $X_\bullet \rightarrow Y_\bullet$ is a simplicial map between proper simplicial spaces that is a levelwise weak equivalence, then $|X_\bullet| \rightarrow |Y_\bullet|$ is a weak equivalence.

Lemma 4.3. *Let A be a topological monoid, B be a right A -module and C be a left A -module. Then if A is well-pointed, $B_\bullet(B, A, C)$ is a good and hence proper simplicial space.*

Proof. This is a consequence of the elementary fact that a product of a cofibration and an identity map is a cofibration. \square

The following is a well-known consequence of Proposition 1.6 of [Seg74]. More generally this theorem is true when the topological monoid A is *group-like*, i.e. if $\pi_0(A)$ is a group.

²Our spaces are always implicitly replaced by their compactly generated weakly Hausdorff replacement, if they are not yet of this type.

Theorem 4.4. *Let A be a well-pointed topological monoid and B be a right A -module. If A is path-connected, then there is a fiber sequence*

$$B \rightarrow B // A \rightarrow * // A$$

An important special case is when $B = A$.

Lemma 4.5. *If A is a well-pointed path-connected topological monoid, then we have that $A // A \simeq *$.*

Proof. Since A has a unit, an extra degeneracy argument implies that the augmentation $|B_\bullet(A, A, *)| \rightarrow *$ is a weak equivalence. \square

Corollary 4.6. *If A is a well-pointed path-connected topological monoid, then the natural map $A \rightarrow \Omega(* // A)$ is a weak equivalence.*

We want to move loops inside geometric realization, which is possible by Theorem 12.3 of [May72].

Theorem 4.7 (May). *If X_\bullet is a proper pointed simplicial space and each X_q is path-connected, then the natural map $|\Omega X_\bullet| \rightarrow \Omega|X_\bullet|$ is a weak equivalence.*

This is an example of commuting homotopy limits and geometric realization. Another such result involves augmented simplicial spaces. If X_\bullet is a simplicial space with augmentation to X_{-1} , for $x \in X_{-1}$ we can form the levelwise homotopy fiber $\text{hofib}_x(X_\bullet \rightarrow X_{-1})$ by defining $\text{hofib}_x(X_p \rightarrow X_{-1})$ to be the space of pairs $(y, \gamma) \in X_p \times \text{Map}([0, 1], X_{-1})$ such that $\gamma(0) = \epsilon(y)$ and $\gamma(1) = x$.

Lemma 4.8. *Let X_\bullet be a proper simplicial space with augmentation to X_{-1} . Then we have that for each $x \in X_{-1}$, $|\text{hofib}_x(X_\bullet \rightarrow X_{-1})| \simeq \text{hofib}_x(|X_\bullet| \rightarrow X_{-1})$.*

Proof. Lemma 2.1 of [RW16] proves this for the thick realizations and since both simplicial spaces are proper by Lemma 4.9, we may replace them with thin realizations. \square

Lemma 4.9. *If X_\bullet is proper, then $\text{hofib}_x(X_\bullet \rightarrow X_{-1})$ is also proper.*

Proof. Write $P_x X_{-1}$ for the space of paths γ in X_{-1} ending at x . In the explicit model for hofib_x given above, we have that $\bigcup_i s_i(\text{hofib}_x(X_p \rightarrow X_{-1})) \cong \text{hofib}_x(\bigcup_i s_i(X_p) \rightarrow X_{-1})$. In particular we have that $\bigcup_i s_i(\text{hofib}_x(X_p \rightarrow X_{-1})) \rightarrow \text{hofib}_x(X_{p+1} \rightarrow X_{-1})$ is the induced map on the pullbacks of rows in

$$\begin{array}{ccccc} P_x X_{-1} & \xrightarrow{\text{ev}_0} & X_{-1} & \xleftarrow{\epsilon} & \bigcup_i s_i(X_p) \\ \parallel & & \parallel & & \downarrow \\ P_x X_{-1} & \xrightarrow{\text{ev}_0} & X_{-1} & \xleftarrow{\epsilon} & X_{p+1} \end{array}$$

As the vertical maps are cofibrations and the left horizontal maps are fibrations, Theorem 1 of [Kie87] implies the induced map on pullbacks is a cofibration. \square

Remark 4.10. The proofs of some of the results in this section rely on notions like quasifibrations. However, for topological and PL-manifolds one needs to work in simplicial

sets. Since simplicial sets and spaces with the Quillen model structures are Quillen equivalent, these theorems hold in simplicial sets if phrased correctly. There is no need for a properness assumption in simplicial sets, see e.g. Proposition IV.1.9 of [GJ09]. Theorem 4.4 is then Corollary 5.8 of [Rez14], while Lemma 4.8 is Proposition 5.4 of [Rez14]. Theorem 4.7 can be replaced by Corollary IV.4.11 of [GJ09].

4.2. Moore versions of diffeomorphism groups. We will define a topological monoid \mathbf{BD} which is a strict model for the H -space $B\mathrm{Diff}_\partial(D^n)$ under boundary connected sum \natural . This is done using a Moore loop-like construction. We then use a similar construction to obtain a module \mathbf{BM} over \mathbf{BD} . This is a strict model for $B\mathrm{Diff}_\partial(M)$ under the action of $B\mathrm{Diff}_\partial(D^n)$ by \natural .

4.2.1. Moore monoid of diffeomorphisms of a disk. We start by defining a topological monoid model for $\mathrm{Diff}_\partial(D^n)$ with boundary connected sum as multiplication. To do this we add a real parameter to constrain the support, and using this parameter to define boundary connected sum by juxtaposition.

Hence we will think of our diffeomorphisms as a subspace of the $[0, \infty) \times \mathrm{Diff}_\partial(D^{n-1} \times [0, \infty))$, with the latter having the topology of C^∞ -convergence on compacts. Even though $[0, \infty) \times \mathrm{Diff}_\partial(D^{n-1} \times [0, \infty))$ is contractible in this topology, our subspace is not.

Definition 4.11. \mathbf{D} is the *Moore monoid of diffeomorphisms of a disk*, given by the subspace of pairs $(t, \phi) \in [0, \infty) \times \mathrm{Diff}_\partial(D^{n-1} \times [0, \infty))$ such that $\mathrm{supp}(\phi) \subset D^{n-1} \times [0, t]$.

The multiplication map is given by $\mathbf{D} \times \mathbf{D} \ni ((t, \phi), (t', \phi')) \mapsto (t + t', \phi \sqcup \phi') \in \mathbf{D}$ with $\phi \sqcup \phi' \in \mathrm{Diff}_\partial(D^{n-1} \times [0, \infty))$ given by

$$(\phi \sqcup \phi')(x, s) = \begin{cases} \phi(x, s) & \text{if } s \leq t \\ (\phi'_1(x, s - t), \phi'_2(x, s - t) + t) & \text{otherwise} \end{cases}$$

and the element $(0, \mathrm{id})$ is the unit.

We start by checking this is indeed a model for $\mathrm{Diff}_\partial(D^n)$.

Lemma 4.12. *The inclusion $\iota: \mathrm{Diff}_\partial(D^{n-1} \times [0, 1]) \hookrightarrow \mathbf{D}$ given by $\phi \mapsto (1, \phi)$ is a homotopy equivalence.*

Proof. We will homotope \mathbf{D} onto $\{1\} \times \mathrm{Diff}_\partial(D^{n-1} \times [0, 1])$ in two steps. In the first step we decrease the size of the support: the pair (t, ϕ) is sent to the path

$$[0, 1] \ni r \mapsto \left(\frac{t}{1+tr}, \phi_r \right) \in \mathbf{D}$$

with ϕ_r given by

$$\phi_r(x, s) = \begin{cases} \left(\phi_1(x, s(1+tr)), \frac{1}{1+tr} \phi_2(x, s(1+tr)) \right) & \text{if } s \in [0, \frac{t}{1+tr}] \\ (x, s) & \text{otherwise} \end{cases}$$

Now we have that $t < 1$, so in the second step we linearly increase t to 1:

$$[0, 1] \ni r \mapsto ((1-r)t + r, \phi) \in \mathbf{D}$$

It is clear that this is homotopic to the identity on $\{1\} \times \mathrm{Diff}_\partial(D^{n-1} \times [0, 1])$. \square

We finish with a technical lemma.

Lemma 4.13. *D is well-pointed.*

Proof. We will give an open neighborhood U of $(0, \text{id})$ which deformation retracts onto $(0, \text{id})$. Pick an open convex neighborhood V of id in the invertible $(n \times n)$ -matrices. The neighborhood U will consist of those pairs (t, ϕ) such that $t < 1$ and for all $(x, s) \in D^{n-1} \times [0, 1]$ the derivative $D_{x,s}(\phi)$ lies in V . The deformation retraction of U onto $(0, \text{id})$ has two steps. The first deforms (t, ϕ) onto (t, id) by linearly interpolating:

$$[0, 1] \ni r \mapsto (t, (1-r) \cdot \phi + r \cdot \text{id}) \in D$$

The inverse function theorem and our choice of V implies this is a path of diffeomorphisms. The second step deforms the pairs (t, id) onto $(0, \text{id})$ by linearly decreasing t :

$$[0, 1] \ni r \mapsto ((1-r)t, \text{id}) \in D$$

□

4.2.2. Moore monoid of classifying spaces of diffeomorphisms of a disk. By Corollary 11.7 of [May72], any unital monoid object in simplicial spaces realizes to a topological monoid. Using this we will produce a topological monoid model for $B\text{Diff}_\partial(D^n)$.

Definition 4.14. There is a simplicial space $[k] \mapsto D_k$ for $k \geq 0$, given by $(k+1)$ -tuples $(t, \phi_1, \dots, \phi_k)$ in $[0, \infty) \times \text{Diff}_\partial(D^{n-1} \times [0, \infty))^k$ such that $\bigcup_i \text{supp}(\phi_i) \subset D^{n-1} \times [0, t]$. The face maps compose diffeomorphisms and the degeneracy maps insert identity. The operations \square described before makes D_\bullet a unital monoid object in simplicial spaces.

We call its geometric realization $|D_\bullet|$ the *Moore monoid of classifying spaces of diffeomorphisms of a disk* and denote it by BD .

We check it has the desired homotopy type.

Lemma 4.15. *The inclusion $B\text{Diff}_\partial(D^{n-1} \times [0, 1]) \hookrightarrow \text{BD}$ given by $x \mapsto (1, x)$ is a weak equivalence.*

Proof. There is a map of simplicial spaces $N_\bullet(\text{Diff}_\partial(D^{n-1} \times [0, 1])) \rightarrow D_\bullet$, which is a levelwise weak equivalence by Lemma 4.12. Since both simplicial spaces are proper by an argument similar to that in the proof of Lemma 4.15, it realizes to a weak equivalence. □

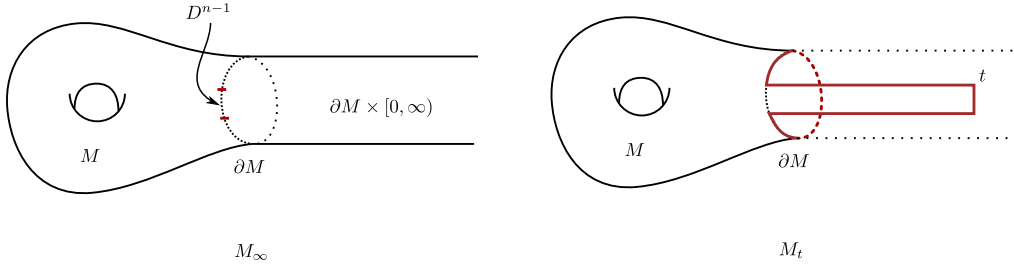
We end with another technical lemma.

Lemma 4.16. *BD is well-pointed.*

Proof. It follows from Lemma 11.3 of [May72] that the geometric realization of a well-pointed simplicial space is well-pointed. It thus suffices to prove that each inclusion $\{(0, \text{id}, \dots, \text{id})\} \hookrightarrow D_k$ is a cofibration. This follows from Lemma 4.13 and the fact that finite products of cofibrations are cofibrations, e.g. by Theorem 1 of [Kie87]. □

4.2.3. Moore modules of diffeomorphisms and classifying spaces of diffeomorphisms of a manifold. We now generalize the definitions of D and BD to an n -dimensional manifold M . To do so, we need an analogue of $D^{n-1} \times [0, \infty)$. Let M_∞ be the non-compact manifold

$$M_\infty := M \cup_{(\partial M \times \{0\})} (\partial M \times [0, \infty))$$

FIGURE 1. An example of M_∞ and M_t .

For $t \in [0, \infty)$ it has a submanifold with corners

$$M_t := M \cup_{(D^{n-1} \times \{0\})} (D^{n-1} \times [0, t])$$

Definition 4.17. \mathbf{M} is the *Moore module of diffeomorphisms of M* . It is given by the subspace of pairs $(t, \phi) \in [0, \infty) \times \text{Diff}(M_\infty)$ such that $\text{supp}(\phi) \subset M_t$.

The right action of \mathbf{D} on \mathbf{M} is given by $\mathbf{M} \times \mathbf{D} \ni ((t, \phi), (t', \phi')) \mapsto (t + t', \phi \sqcup \phi') \in \mathbf{M}$ with $\phi \sqcup \phi'$ given by

$$(\phi \sqcup \phi')(m) = \begin{cases} \phi(m) & \text{if } m \in M_t \\ (\phi'_1(x, s - t), \phi'_2(x, s - t) + t) & \text{if } m = (x, s) \in D^{n-1} \times [t, \infty) \\ m & \text{otherwise} \end{cases}$$

A similar proof as for Corollary 11.7 of [May72] implies that a module object over a unital monoid object in simplicial spaces realizes to a module over the topological monoid.

Definition 4.18. There is a simplicial space $[k] \mapsto \mathbf{M}_k$ for $k \geq 0$, given by $(k + 1)$ -tuples $(t, \phi_1, \dots, \phi_k)$ in $[0, \infty) \times \text{Diff}(M_\infty)^k$ such that $\bigcup_i \text{supp}(\phi_i) \subset M_t$. The operations \sqcup described before makes it a module object in simplicial spaces over the unital monoid object \mathbf{D}_\bullet .

We call its geometric realization $|\mathbf{M}_\bullet|$ the *Moore module of classifying spaces of diffeomorphisms of M* and denote it by \mathbf{BM} .

A similar proof as for Lemma's 4.12, 4.13 and 4.15 gives the following.

Lemma 4.19. *The simplicial space \mathbf{M}_\bullet is proper. The maps $\text{Diff}_\partial(M) \rightarrow \mathbf{M}$ given by $\phi \mapsto (0, \phi)$ and $B\text{Diff}_\partial(M) \rightarrow \mathbf{BM}$ given by $x \mapsto (0, x)$ are weak equivalences.*

4.3. A delooped fiber sequence. In this subsection our goal is to construct the fiber sequence (2)

$$B\text{Diff}_\partial(M) \rightarrow B\text{Emb}_{1/2\partial}^\cong(M) \rightarrow B(B\text{Diff}_\partial(D^n), \mathfrak{d})$$

The precise statement is as follows, and the proof is given in the remainder of this subsection.

Theorem 4.20. *There is a fiber sequence*

$$\mathbf{BM} \rightarrow \mathbf{BM} // \mathbf{BD} \rightarrow * // \mathbf{BD}$$

with $\mathbf{BM} \simeq \mathbf{BDiff}_\partial(M)$, $\mathbf{BM} // \mathbf{BD} \simeq \mathbf{BEmb}_{1/2\partial}^\simeq(M)$ and $\Omega(* // \mathbf{BD}) \simeq \mathbf{BDiff}_\partial(D^n)$. The map $\mathbf{BM} \rightarrow \mathbf{BM} // \mathbf{BD}$ is weakly equivalent to the inclusion $\mathbf{BDiff}_\partial(M) \hookrightarrow \mathbf{BEmb}_{1/2\partial}^\simeq(M)$.

Proof. This follows from Theorem 4.4, because the monoid \mathbf{BD} is path-connected. The identifications of fiber, total space and base are Lemma 4.19, and Propositions 4.31 and 4.21 respectively. The statement about the map $\mathbf{BM} \rightarrow \mathbf{BM} // \mathbf{BD}$ is Lemma 4.30. \square

4.3.1. *The base $* // \mathbf{BD}$.* We start by describing the base, by showing that $* // \mathbf{BD}$ is indeed a delooping of $\mathbf{BDiff}_\partial(D^n)$.

Proposition 4.21. *We have a weak equivalence*

$$\mathbf{BDiff}_\partial(D^n) \simeq \Omega(* // \mathbf{BD})$$

Proof. Since \mathbf{BD} is path-connected, the map $\mathbf{BD} \rightarrow \Omega(* // \mathbf{BD})$ is a weak equivalence by Corollary 4.6. By Lemma 4.15 the map $\mathbf{BDiff}_\partial(D^{n-1} \times [0, 1]) \rightarrow \mathbf{BD}$ is a weak equivalence. Finally it is standard that $\mathbf{BDiff}_\partial(D^{n-1} \times [0, 1]) \simeq \mathbf{BDiff}_\partial(D^n)$. \square

Remark 4.22. In fact, $\mathbf{BDiff}_\partial(D^n)$ is an n -fold loop space. To see this, note it is an algebra over the little n -disks operad E_n . Since it is path-connected, May's recognition principle says it is equivalent to an n -fold loop space as an E_n -algebra [May72]. Smoothing theory as in Theorem 5.12 gives us a particular n -fold delooping: $\mathbf{BDiff}_\partial(D^n)$ is equivalent to the identity component of $\Omega^n \mathbf{PL}(n)/O(n)$. One can replace $\mathbf{PL}(n)$ with $\mathbf{Top}(n)$ if $n \neq 4$.

4.3.2. *Restricting the images of submanifolds.* On our way to proving that $\mathbf{BM} // \mathbf{BD} \simeq \mathbf{BEmb}_{1/2\partial}^\simeq(M)$, it will be useful to study diffeomorphisms and embeddings with a restriction on the images of certain submanifolds. In particular we will define weakly equivalent subspaces $\mathbf{M}^{(\text{im})} \subset \mathbf{M}$ and $E^{(\text{im})} \subset \mathbf{Emb}_{1/2\partial}^\simeq(M)$ with the advantage that there is a well-behaved restriction map $\mathbf{M}^{(\text{im})} \rightarrow E^{(\text{im})}$.

We start with the case of diffeomorphisms.

Definition 4.23. Let $\mathbf{Diff}^{(\text{im})}(M_\infty)$ be the subspace of $\mathbf{Diff}(M_\infty)$ consisting of those diffeomorphisms φ such that $\varphi(M_0) \subset M_0$.

Using this we can define $\mathbf{M}^{(\text{im})}$ as the subspace of \mathbf{M} consisting of $(t, \varphi) \in [0, \infty) \times \mathbf{Diff}(M_\infty)$ such that $\varphi \in \mathbf{Diff}^{(\text{im})}(M_\infty)$. The \mathbf{D} -module structure on \mathbf{M} restricts to a \mathbf{D} -module structure on $\mathbf{M}^{(\text{im})}$.

Lemma 4.24. *The inclusion $\mathbf{M}^{(\text{im})} \hookrightarrow \mathbf{M}$ is a weak equivalence of \mathbf{D} -modules.*

Proof. It suffices to push the image of M_0 under a family of diffeomorphisms out of $\text{int}(D^{n-1}) \times (0, \infty)$. To do this, let \mathbf{M}' be the subspace of \mathbf{M} consisting of those pairs (t, φ) that satisfy the properties that $t \leq 1/2$ and that φ is the identity on $(D^{n-1} \setminus \frac{1}{2}D^{n-1}) \times (0, \infty)$. Equivalently, these are the diffeomorphisms ϕ supported in $M_0 \cup (\frac{1}{2}D^{n-1} \times [0, 1])$. In particular $\phi(M_0) \subset M_0 \cup (\frac{1}{2}D^{n-1} \times [0, 1/2])$. The inclusion $\mathbf{M}' \hookrightarrow \mathbf{M}$ is a weak equivalence by an argument similar to that of Lemma 4.12.

Now pick a family of compactly-supported diffeomorphisms $\psi_t: M_\infty \rightarrow M_\infty$ for $t \in [0, 1]$ with the following properties:

- (i) ψ_t is the identity on $(\partial M \setminus \text{int}(D^{n-1})) \times [0, \infty)$ and $D^{n-1} \times [1, \infty)$,

- (ii) $\psi_0 = \text{id}$,
- (iii) $\psi_t(M_0 \cup (\frac{1}{2}D^{n-1} \times [0, t])) \subset M_0$

The diffeomorphism $\psi_t \phi \psi_t^{-1}$ has support $\psi_t(\text{supp}(\phi))$. Hence conjugating with ψ_t gives us a homotopy starting at the identity map of M which deforms the subspace M' into diffeomorphisms with support in M_0 , by condition (iii) for $t = 1$. This implies they must send M_0 into M_0 , hence these diffeomorphisms lie in $M^{(\text{im})}$. By condition (iii) for all $t \in [0, 1]$, this homotopy preserves the subspace $M^{(\text{im})}$, so we conclude that $M^{(\text{im})} \hookrightarrow M$ is a weak equivalence. \square

Next we describe the relevant spaces of embeddings. The end result will be three variants

$$\text{Emb}_{1/2\partial}^{\cong}(M) \hookrightarrow E^{(\text{im})} \hookrightarrow \text{Emb}_{1/2\partial}^{\cong}(M'_{\infty}, M_{\infty})$$

such that the inclusions are weak equivalences.

Let M'_{∞} be the submanifold of M_{∞} given by

$$M \cup_{(\partial M \setminus \text{int}(D^{n-1})) \times \{0\}} (\partial M \setminus \text{int}(D^{n-1})) \times [0, \infty)$$

Let us consider an embedding e of M'_{∞} into M_{∞} that is the identity on $(\partial M \setminus \text{int}(D^{n-1})) \times [0, \infty)$. The closure of the complement of $\text{im}(e)$ will be manifold with corners, denoted $C(e)$. Its boundary is identified with $D^{n-1} \times [0, \infty)$ by e . Thus we can demand that $C(e)$ is diffeomorphic to $D^{n-1} \times [0, \infty)$ by a diffeomorphism ϕ that preserves the identification of the boundary. Furthermore, since $\text{im}(e)$ is compact, it also makes sense for this diffeomorphism to have compact support.

Definition 4.25. Let $\text{Emb}_{1/2\partial}^{\cong}(M'_{\infty}, M_{\infty})$ be the space of embeddings $e: M'_{\infty} \hookrightarrow M_{\infty}$ that are the identity on $(\partial M \setminus \text{int}(D^{n-1})) \times [0, \infty)$ and such that $C(e)$ admits a compactly supported diffeomorphism to $D^{n-1} \times [0, \infty)$ rel boundary.

Definition 4.26. Let $E^{(\text{im})}$ be the subspace of $\text{Emb}_{1/2\partial}^{\cong}(M'_{\infty}, M_{\infty})$ consisting of embeddings e that satisfy $e(M_0) \subset M_0$.

A proof similar to that of Lemma 4.24 gives the following:

Lemma 4.27. *The inclusion $E^{(\text{im})} \hookrightarrow \text{Emb}_{1/2\partial}^{\cong}(M'_{\infty}, M_{\infty})$ is a weak equivalence.*

We extend an embedding $e \in \text{Emb}_{1/2\partial}(M)$ to an embedding $\bar{e} \in \text{Emb}_{1/2\partial}(M'_{\infty}, M_{\infty})$ by defining it to be the identity on $(\partial M \setminus \text{int}(D^{n-1})) \times [0, \infty)$.

Lemma 4.28. *Extension induces an inclusion*

$$\text{Emb}_{1/2\partial}^{\cong}(M) \hookrightarrow \text{Emb}_{1/2\partial}^{\cong}(M'_{\infty}, M_{\infty})$$

which is a weak equivalence.

Proof. Since the inclusion factors through the embeddings e such that $e(M_0) \subset M_0$, it suffices to show that $\text{Emb}_{1/2\partial}^{\cong}(M) \hookrightarrow E^{(\text{im})}$ is a weak equivalence. To do so it suffices to check that e is isotopic to a diffeomorphism if and only if its extension \bar{e} has complement whose closure $C(e)$ admits a compactly-supported diffeomorphism to $D^{n-1} \times [0, \infty)$ rel boundary. For the direction \Rightarrow , we note that the condition only depends on the isotopy class of e and hence we may assume e is a diffeomorphism that fixes the boundary. Then C is exactly $D^{n-1} \times [0, \infty)$.

For the direction \Leftarrow , pick a compactly-supported diffeomorphism $\theta: D^{n-1} \times [0, \infty) \rightarrow C(e)$ rel boundary. By rescaling we can assume the support is contained in $M_0 \cup (D^{n-1} \times [0, 1])$. We can extend this by the identity to a compactly-supported diffeomorphism

$$\vartheta: \partial M \times [0, \infty) \rightarrow C(e) \cup (\partial M \setminus \text{int}(D^{n-1})) \times [0, \infty)$$

We will use ϑ to isotope e to an extension $\bar{\phi}$ of a diffeomorphism ϕ of M fixing the boundary. In words, this isotopy pulls the non-trivial part of $C(e)$ into a collar. Pick an inner collar $\partial M \times [-2, 0] \rightarrow M$ and let $\lambda_t: [-2, 0] \rightarrow [-2, 0]$ be a family of embeddings with the following properties:

- (i) $\lambda_0 = \text{id}$,
- (ii) $\lambda_t(-2) = -2$, and
- (iii) λ_t near 0 is translation to $-t$.

Then the path e_t of embeddings for $t \in [0, 1]$, defined as

$$e_t(m) = \begin{cases} e(m) & \text{if } m \in M'_\infty \setminus (\partial M \times [-2, 0]) \\ (\text{id} \times \lambda_t)(e(m', \lambda_t^{-1}(s))) & \text{if } m = (m', s) \in \partial M \times [-2, -t] \\ (\vartheta_1(m', s+t), \vartheta_1(m', s+t) - t) & \text{if } m = (m', s) \in \partial M \times (-t, 0] \end{cases}$$

is an isotopy in $\text{Emb}_{1/2\partial}^\cong(M'_\infty, M_\infty)$ from e to an extension of a diffeomorphism. In particular, e_1 coincides with e on $M'_\infty \setminus (\partial M \times [-2, 0])$, while on $(\partial M \times [-2, 0])$ its image coincides with the union of $\lambda_1(e(\partial M \times [-2, 0]))$ and $\vartheta(\partial M \times [0, 1])$ shifted by -1 in the second coordinate. Since ϑ has support in $M_0 \cup (D^{n-1} \times [0, 1])$ this is exactly M_0 . Finally note that in the previous lemma we showed $E^{(\text{im})} \rightarrow \text{Emb}_{1/2\partial}^\cong(M'_\infty, M_\infty)$ is a weak equivalence, so that e is also isotopic in $E^{(\text{im})}$ to an extension of a diffeomorphism. \square

Restriction to M_0 gives the vertical maps in a commutative diagram

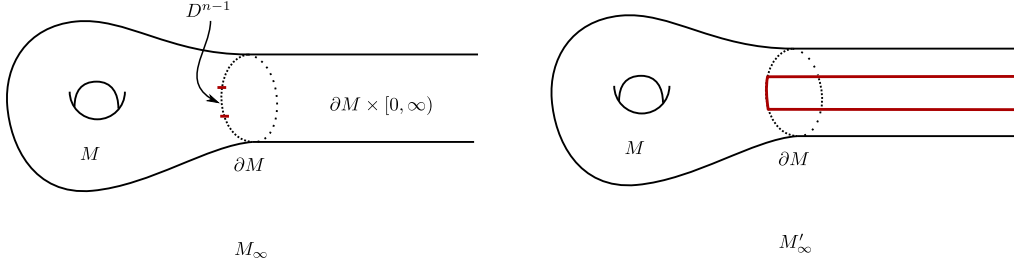
$$\begin{array}{ccc} \mathbf{M}^{(\text{im})} & \xrightarrow{\simeq} & \mathbf{M} \\ \downarrow & & \downarrow \\ \text{Emb}_{1/2\partial}^\cong(M) & \xrightarrow[\simeq]{} & \text{Emb}_{1/2\partial}^\cong(M'_\infty, M_\infty) \end{array}$$

As mentioned before, the value of considering the left map is that if both $\mathbf{M}^{(\text{im})}$ and $\text{Emb}_{1/2\partial}^\cong(M)$ are given the topological monoid structure coming from composition, it is a map of topological monoids. This will be used in the next subsection.

4.3.3. The total space. We will now finish the proof that $\mathbf{BM} \parallel \mathbf{BD} \simeq B\text{Emb}_{1/2\partial}^\cong(M)$. To do so, we will construct a map $\mathbf{BM}^{(\text{im})} \parallel \mathbf{BD} \rightarrow B\text{Emb}_{1/2\partial}^\cong(M)$. Here $\mathbf{BM}^{(\text{im})}$ is constructed as the realization of the simplicial subspace $\mathbf{M}_\bullet^{(\text{im})}$ of \mathbf{M}_\bullet with k -simplices given by $(p+1)$ -tuples $(t, \phi_1, \dots, \phi_k)$ in $[0, \infty) \times \text{Diff}^{(\text{im})}(M_\infty)^k$.

Lemma 4.29. *We have that the inclusion $\mathbf{BM}^{(\text{im})} \hookrightarrow \mathbf{BM}$ is a weak equivalence.*

Proof. Since both simplicial spaces are proper, it suffices to show that $\mathbf{M}_\bullet^{(\text{im})} \rightarrow \mathbf{M}_\bullet$ is a levelwise equivalence. The lemma follows by noting that the inclusion $\mathbf{M}_k^{(\text{im})} \rightarrow \mathbf{M}_k$ is weakly equivalent to the k -fold product of the inclusion $\mathbf{M}^{(\text{im})} \hookrightarrow \mathbf{M}$, which is a weak equivalence by Lemma 4.24. \square

FIGURE 2. An example of M_∞ and M'_∞ .

Next we remark that restriction to M_0 induces a simplicial map $\mathbf{M}_\bullet^{(\text{im})} \rightarrow N_\bullet(\text{Emb}_{1/2\partial}^\cong(M))$, which realizes to a map

$$\mathbf{BM}^{(\text{im})} \rightarrow B\text{Emb}_{1/2\partial}^\cong(M)$$

As before, BD-module structure on \mathbf{BM} restricts to $\mathbf{BM}^{(\text{im})}$, and this is a map BD-modules if $B\text{Emb}_{1/2\partial}^\cong(M)$ is given the trivial BD-module structure. Take the homotopy quotient by BD, followed by projection, to obtain a map

$$\eta: \mathbf{BM}^{(\text{im})} // \text{BD} \rightarrow B\text{Emb}_{1/2\partial}^\cong(M) \times * // \text{BD} \rightarrow B\text{Emb}_{1/2\partial}^\cong(M)$$

The following lemma and proposition establish two important properties of this map.

Lemma 4.30. *The composition*

$$B\text{Diff}_\partial(M) \hookrightarrow \mathbf{BM}^{(\text{im})} \hookrightarrow \mathbf{BM}^{(\text{im})} // \text{BD} \xrightarrow{\eta} B\text{Emb}_{1/2\partial}^\cong(M)$$

where the first map is as in Lemma 4.19 and the second is the inclusion of 0-simplices, coincides with the inclusion $B\text{Diff}_\partial(M) \hookrightarrow B\text{Emb}_{1/2\partial}^\cong(M)$.

Proof. By inspection. □

Proposition 4.31. *The map*

$$\mathbf{BM}^{(\text{im})} // \text{BD} \xrightarrow{\eta} B\text{Emb}_{1/2\partial}^\cong(M)$$

is a weak equivalence.

Proof. Since both the domain and target are path-connected, it suffices to show that the induced map on based loop spaces is a weak equivalence. Recall $\mathbf{BM}^{(\text{im})} // \text{BD}$ is the realization of the bar construction $B_\bullet(\mathbf{BM}^{(\text{im})}, \text{BD}, *)$. This is a proper simplicial space by Lemma 4.16 and has path-connected spaces of p -simplices. Thus by Theorem 4.7 the map

$$|\Omega(B_\bullet(\mathbf{BM}^{(\text{im})}, \text{BD}, *))| \rightarrow \Omega|B_\bullet(\mathbf{BM}^{(\text{im})}, \text{BD}, *)| = \Omega(\mathbf{BM}^{(\text{im})} // \text{BD})$$

is a weak equivalence. The inclusion of simplicial spaces

$$\epsilon: B_\bullet(\mathbf{M}^{(\text{im})}, \text{D}, *) \rightarrow \Omega(B_\bullet(\mathbf{BM}^{(\text{im})}, \text{BD}, *))$$

is a levelwise weak equivalence. Since both simplicial spaces are proper, this map induces a weak equivalence on geometric realization. We conclude it suffices to show that

$$M^{(\text{im})} // D \rightarrow \text{Emb}_{1/2\partial}^{\cong}(M)$$

is a weak equivalence. This map fits into a commutative diagram

$$\begin{array}{ccc} M^{(\text{im})} // D & \xrightarrow{\cong} & M // D \\ \epsilon \downarrow & & \downarrow \epsilon \\ \text{Emb}_{1/2\partial}^{\cong}(M) & \xrightarrow{\cong} & \text{Emb}_{1/2\partial}^{\cong}(M'_{\infty}, M_{\infty}) \end{array}$$

with horizontal maps weak equivalences by Lemma's 4.13, 4.24, and 4.28. The right map is a weak equivalence by Lemma 4.32. \square

Lemma 4.32. *The map $\epsilon: M // D \rightarrow \text{Emb}_{1/2\partial}^{\cong}(M'_{\infty}, M_{\infty})$ is a weak equivalence.*

Proof. Consider the submodule M_1 of M consisting of pairs (t, x) with $t \geq 1$. We claim the inclusion $M_1 \hookrightarrow M$ is a weak equivalence. To see this, note that it is in fact a deformation retract, with deformation retraction given by linearly increasing t of an element $(t, \phi) \in M$ to a number ≥ 1 :

$$[0, 1] \ni r \mapsto (\min(r, t), \phi) \in M$$

The map induced by restriction to M_0 can be interpreted as an augmentation map

$$\epsilon: |B_{\bullet}(M_1, D, *)| \rightarrow \text{Emb}_{1/2\partial}^{\cong}(M'_{\infty}, M_{\infty})$$

More precisely, this map is on p -simplices $M_1 \times D^p$ given by projecting away the term D^p and then applying the map $M_1 \rightarrow \text{Emb}_{1/2\partial}^{\cong}(M'_{\infty}, M_{\infty})$ given by restriction to $M_0 \subset M_{\infty}$.

It suffices to show ϵ has weakly contractible homotopy fibers. We start by identifying the homotopy fibers using Lemma 4.8. Since $B_{\bullet}(M_1, D, *)$ is proper, for all $e \in \text{Emb}_{1/2\partial}^{\cong}(M'_{\infty}, M_{\infty})$ we have that

$$|\text{hofib}_e(B_{\bullet}(M_1, D, *) \rightarrow \text{Emb}_{1/2\partial}^{\cong}(M'_{\infty}, M_{\infty}))| \simeq \text{hofib}_e(|B_{\bullet}(M_1, D, *)| \rightarrow \text{Emb}_{1/2\partial}^{\cong}(M'_{\infty}, M_{\infty}))$$

Fixing $p \geq 0$, the map $B_p(M_1, D, *) = M_1 \times D^p \rightarrow \text{Emb}_{1/2\partial}^{\cong}(M'_{\infty}, M_{\infty})$ is given by the composition of first projecting away the terms D^p and the restriction map $M_1 \rightarrow \text{Emb}_{1/2\partial}^{\cong}(M'_{\infty}, M_{\infty})$. Projection is a fibration, so to prove the composite is a fibration it suffices to prove the restriction map is a fibration. This will follow from the isotopy extension theorem. Suppose we are given for $i \geq 0$ some diagram

$$\begin{array}{ccc} D^i & \xrightarrow{g} & M_1 \\ \downarrow & & \downarrow \epsilon \\ D^i \times [0, 1] & \xrightarrow{G} & \text{Emb}_{1/2\partial}^{\cong}(M'_{\infty}, M_{\infty}) \end{array}$$

Write the top map as $(\tau, \gamma): D^i \rightarrow [0, \infty) \times \text{Diff}(M_{\infty})$. There is a continuous map

$$\Theta: \text{Emb}_{1/2\partial}^{\cong}(M'_{\infty}, M_{\infty}) \rightarrow [0, \infty)$$

recording for an embedding e the minimal value of t such that embedding e has image contained in $M_t \cup M'_{\infty}$. We remark that $\tau > \Theta \circ G|_{D^i}$, because we used M_1 and so the image

of M_0 is not only contained in M_τ but in fact has an open neighborhood in M_τ (if we had used \mathbf{M} we only would have \geq). Since D^i is compact, we can find a $\delta > 0$ and a continuous function $T: D^i \times [0, 1] \rightarrow [1, \infty)$ such that (a) $T|_{D^i} = \tau$ and (b) $T > \Theta \circ G + \delta$ on $D^i \times [0, 1]$.

Recall isotopy extension for embeddings is proven by an argument that in essence amounts to taking the derivative of a family of embeddings, extending this to a time-dependent vector field on the entire manifold and flowing along it. We bring this up, because it implies we can control the support of our isotopies and find a map $\Psi: D^n \times [0, 1] \rightarrow \text{Diff}_c(M_\infty)$ such that (a) $\Psi(x, 0) = \text{id}$ and $\Psi(x, s) \circ G(x, 0) = G(x, s)$ and (b) $\text{supp}(\Psi(x, s)) \subset M_{\Theta \circ G(x, s) + \delta}$.

Then our lift is given by

$$\begin{aligned} L: D^i \times [0, 1] &\rightarrow \mathbf{M}_1 \subset [0, \infty) \times \text{Diff}(M_\infty) \\ (x, s) &\mapsto (T(x, s), G(x, s) \circ g(x)) \end{aligned}$$

The condition (a) on T and Ψ guarantees this is a lift, while the condition (b) implies that $\text{supp}(G(x, s) \circ g(x)) \subset M_{T(x, s)}$.

As a consequence of proving that these maps are fibrations, we can replace the levelwise homotopy fiber $|\text{hofib}_e(B_\bullet(\mathbf{M}_1, \mathbf{D}, *) \rightarrow \text{Emb}_{1/2\partial}^\cong(M'_\infty, M_\infty))|$ with the levelwise fiber $\epsilon^{-1}(e)_\bullet$. This satisfies $|\epsilon^{-1}(e)_\bullet| \cong |B_\bullet(\mathbf{M}_{e,1}, \mathbf{D}, *)|$ with $\mathbf{M}_{e,1}$ be the subspace of \mathbf{M}_1 of diffeomorphisms that agree with e on M_0 .

Hence it suffices to prove that $|B_\bullet(\mathbf{M}_{e,1}, \mathbf{D}, *)|$ is weakly contractible. Since homotopy fibers only depend on the path component of the base point, it suffices to check this only for a particular point in each path component.

Note that the image of any element of $\text{Diff}_\partial(M)$ in $\text{Emb}_{1/2\partial}^\cong(M'_\infty, M_\infty)$ is the identity on ∂M_0 . By construction $\text{Diff}_\partial(M) \rightarrow \text{Emb}_{1/2\partial}^\cong(M'_\infty, M_\infty)$ is surjective on path components — this is the reason for including the superscript \cong — and thus in each component we can find an embedding e that is equal to the identity on ∂M_0 , from which it also follows that $\text{im}(e) = M_0$. In that case, let \mathbf{D}_1 denote the subspace of \mathbf{D} of pairs (t, ϕ) such that $t \geq 1$. Then we have that $|B_\bullet(\mathbf{M}_{e,1}, \mathbf{D}, *)|$ is homeomorphic to $|B_\bullet(\mathbf{D}_1, \mathbf{D}, *)|$ and thus weakly equivalent to $|B_\bullet(\mathbf{D}, \mathbf{D}, *)| = \mathbf{D} // \mathbf{D}$, which is weakly contractible by Lemma 4.5. \square

4.4. Generalizations. We state three variations to Theorem 4.20 without proof, as the required modifications are straightforward.

4.4.1. Identity components. The group of isotopy classes of diffeomorphisms of a path-connected manifold with boundary has an action of $\pi_0(\text{Diff}_\partial(D^n)) \cong \Theta_{n+1}$. This is given by taking the boundary connected sum of representatives and re-identifying the resulting manifold $D^n \natural M$ with M . This operation is also denoted by \natural . Using it we define the second inertia group of M as in [Lev70].

Definition 4.33. The *second inertia group* of M is the subgroup of $\pi_0(\text{Diff}_\partial(D^n)) \cong \Theta_{n+1}$ of isotopy classes h with the property that $h \natural \text{id}_M$ is isotopic to id_M , i.e. the stabilizer of id_M .

Definition 4.34. We now define variations on the various spaces defined earlier:

- Let $\text{Emb}_{1/2\partial}^{\text{id}}(M)$ denote the identity component of $\text{Emb}_{1/2\partial}(M)$ and $\text{Diff}_\partial^{\text{id}}(M)$ the identity component of $\text{Diff}_\partial(M)$.

- Let \mathbf{BM}_{id} be the subspace of \mathbf{BM} where all diffeomorphisms lie in the identity component.
- If H is a subgroup of $\pi_0(\text{Diff}_{\partial}(D^n))$, let $\text{Diff}_{\partial}^H(D^n)$ denote the subgroup of $\text{Diff}_{\partial}(D^n)$ consisting of those connected components.
- If H is a subgroup of $\pi_0(\text{Diff}_{\partial}(D^n))$, let \mathbf{BD}^H denote the subspace of \mathbf{BD} where all diffeomorphisms lie in H .

Corollary 4.35. *Let $H \subset \pi_0(\text{Diff}_{\partial}(D^n))$ be the second inertia group of M . There is a fiber sequence*

$$\mathbf{BM}^{\text{id}} \rightarrow \mathbf{BM}^{\text{id}} // \mathbf{BD}^H \rightarrow * // \mathbf{BD}^H$$

with $\mathbf{BM}^{\text{id}} \simeq \mathbf{BDiff}_{\partial}^{\text{id}}(M)$, $\mathbf{BM}^{\text{id}} // \mathbf{BD}^H \simeq \mathbf{BEmb}_{1/2\partial}^{\text{id}}(M)$ and $\Omega(* // \mathbf{BD}^H) \simeq \mathbf{BDiff}_{\partial}^H(D^n)$.

4.4.2. Setwise fixed subsets of the boundary. Our next generalization concerns diffeomorphisms and embeddings that fix a submanifold A of the boundary setwise, instead of pointwise. Let M be an n -dimensional manifold with boundary ∂M , $A \subset \partial M$ a codimension zero submanifold and $D^{n-1} \subset \partial M \setminus \text{int}(A)$ an embedded disk.

Definition 4.36. We now define some variations on the various spaces used before:

- Let $\text{Diff}_{\partial,A}(M)$ be the diffeomorphisms that are the identity on $\partial M \setminus A$ and fix A setwise.
- Let $\text{Emb}_{1/2\partial,A}(M)$ be the self-embeddings of M that are the identity on $\partial M \setminus (\text{int}(A) \cup \text{int}(D^{n-1}))$ and fix A setwise.
- Let $\text{Emb}_{1/2\partial,A}^{\cong}(M)$ be the self-embeddings of M that are the identity on $\partial M \setminus (\text{int}(A) \cup \text{int}(D^{n-1}))$, fix A setwise, and are isotopic through embeddings satisfying these conditions to a diffeomorphism that is the identity on $\partial M \setminus \text{int}(A)$ and fixes A setwise.

Corollary 4.37. *There is a fiber sequence*

$$\mathbf{BM}_A \rightarrow \mathbf{BM}_A // \mathbf{BD} \rightarrow * // \mathbf{BD}$$

with $\mathbf{BM}_A \simeq \mathbf{BDiff}_{\partial,A}(M)$, $\mathbf{BM}_A // \mathbf{BD} \simeq \mathbf{BEmb}_{1/2\partial,A}^{\cong}(M)$ and $\Omega(* // \mathbf{BD}) \simeq \mathbf{BDiff}_{\partial}(D^n)$.

Let $H \subset \pi_0(\text{Diff}_{\partial}(D^n))$ be the second inertia group of M . There is a fiber sequence

$$\mathbf{BM}_A^{\text{id}} \rightarrow \mathbf{BM}_A^{\text{id}} // \mathbf{BD}^H \rightarrow * // \mathbf{BD}^H$$

with $\mathbf{BM}_A^{\text{id}} \simeq \mathbf{BDiff}_{\partial}^{\text{id}}(M)$, $\mathbf{BM}_A^{\text{id}} // \mathbf{BD}^H \simeq \mathbf{BEmb}_{1/2\partial,A}^{\text{id}}(M)$ and $\Omega(* // \mathbf{BD}^H) \simeq \mathbf{BDiff}_{\partial}^H(D^n)$.

4.4.3. Homeomorphisms and PL-homeomorphisms. The final generalization concerns other categories of manifolds: $\text{CAT} = \text{Top}, \text{PL}$. In this case the Alexander trick tells us that $\text{CAT}_{\partial}(D^n) \simeq *$, so the fiber sequences become weak equivalences. We remark that the relevant versions of isotopy extension hold in these categories as well.

Corollary 4.38. *There are weak equivalences*

$$\mathbf{BCAT}_{\partial,A}(M) \simeq \mathbf{BEmb}_{1/2\partial,A}^{\text{CAT}}(M) \quad \text{and} \quad \mathbf{BCAT}_{\partial,A}^{\text{id}}(M) \simeq \mathbf{BEmb}_{1/2\partial,A}^{\text{CAT},\text{id}}(M)$$

One subtlety here is that we need to work in simplicial sets and invoke Remark 4.10 to make sure the technical tools still apply. The reason for using simplicial sets is twofold: firstly, there is no reasonable topology on PL-homeomorphisms or PL-embeddings. Secondly,

in both PL and Top, all embeddings and families of embeddings should be locally flat for the isotopy extension theorem to be true. This is not a pointwise condition, so requires the use of simplicial sets.

5. PROOFS OF MAIN RESULTS

In this section we prove the results announced in the introduction, summarized as follows:

- (5.1) $B\text{Diff}_\partial(D^{2n})$. This uses the fiber sequence (2). We understand the total space using embedding calculus in the form of Theorem 3.2 and the fiber using the results of Galatius and Randal-Williams.
- (5.2) $B\text{Diff}_\partial(M)$ for $\dim(M) = 2n$. This uses the fiber sequence (1). We understand the base using embedding calculus and the fiber using the information obtained about $B\text{Diff}_\partial(D^{2n})$.
- (5.3) $B\text{Diff}_\partial(D^{2n+1})$ and $B\text{Diff}_\partial(M)$ for $\dim(M) = 2n + 1$. These arguments are similar to those in Subsection 5.1 and 5.2. One replaces the results of Galatius and Randal-Williams with those of Botvinnik and Perlmutter for odd-dimensional manifolds, but to apply Theorem 3.2 it turns out we need some results for even-dimensional manifolds obtained in Subsection 5.2.
- (5.4) $B\text{Top}(n)$ and $B\text{PL}(n)$. Using smoothing theory, we relate $\text{Top}(n)$ and $\text{PL}(n)$ to diffeomorphisms of disks and we apply the results obtained in Subsections 5.1 and 5.3.
- (5.5) $B\text{Top}_\partial(M)$ and $B\text{PL}_\partial(M)$. These are studied using embedding calculus and the information obtained about $B\text{Top}(n)$ and $B\text{PL}(n)$ in Subsection 5.4.
- (5.6) $BC(D^n)$, $\text{Wh}^{\text{Diff}}(*)$ and $A(*)$. These follow from the results in Subsections 5.1 and 5.3.
- (5.7) $B\text{haut}(M)$ and $\widetilde{B\text{CAT}}(M)$. These are studied independently of the previous results. We also treat $\widetilde{\text{CAT}}(M)/\text{CAT}(M)$.

5.1. Diffeomorphisms of the even-dimensional disk. Let $\#$ denote connected sum. The manifolds

$$W_{g,1} := (\#_g S^n \times S^n) \setminus \text{int}(D^{2n})$$

play an important role in the next proof.

Theorem 5.1. *Let $2n \neq 4$, then $B\text{Diff}_\partial(D^{2n})$ is in Fin . It is thus in particular of homotopically and homologically finite type.*

Proof. The case $2n = 2$ follows from [Sma59b], so we restrict our attention to $2n \geq 6$. Let us prove that $* \parallel \text{BD} \in \text{HFin}$. Consider the fiber sequence of Theorem 4.20 in the case $M = W_{g,1}$:

$$\text{BW}_{g,1} \rightarrow \text{BW}_{g,1} \parallel \text{BD} \rightarrow * \parallel \text{BD}$$

We want to apply part (i) for bases of Lemma 2.5 and to do so, it suffices to prove that for fixed N , the homology groups $H_*(\text{BW}_{g,1} \parallel \text{BD})$ and $H_*(\text{BW}) \cong H_*(B\text{Diff}_\partial(W_{g,1}))$ are finitely generated for $* \leq N$ if g is sufficiently large.

On the one hand, in [GRW14a, GRW14b] Galatius and Randal-Williams proved that if $* \leq \frac{g-3}{2}$ we have an isomorphism

$$H_*(B\text{Diff}_\partial(W_{g,1})) \cong H_*(\Omega_0^\infty(MT\theta))$$

Here $MT\theta$ is the Thom spectrum of $-\theta^*\gamma$, with $\theta: BO(2n)\langle n \rangle \rightarrow BO(2n)$ the n -connective cover, γ the universal vector bundle over $BO(2n)$, and Ω_0^∞ denotes a component of the infinite loop space. A component of an infinite loop space of a spectrum is of homotopically finite type if the spectrum has finitely generated homotopy groups in positive degrees. A bounded-below spectrum has finitely generated homotopy groups if and only if it has finitely generated homology groups. This is true for $MT\theta$ since the Thom isomorphism says that its homology is a shift of the homology of $BO(2n)\langle n \rangle$, which is in \mathbf{Fin} by Example 2.19.

On the other hand, by Proposition 4.31, $BW_{g,1} // BD \simeq B\text{Emb}_{1/2\partial}^\infty(W_{g,1})$ and thus by Theorem 3.2, we have that $BW_{g,1} // BD \in \mathbf{HFin}$ for all g .

By part (i) for bases of Lemma 2.5 we conclude $* // BD \in \mathbf{HFin}$. Since BD is path-connected, $* // BD$ is simply-connected and by Lemma 2.17 we can conclude that $* // BD \in \mathbf{Fin}$. Proposition 4.21 says $\Omega(* // BD) \simeq \text{Diff}_\partial(D^{2n})$, and the latter is in $\mathbf{\Pi Fin}$. But because we know $\pi_0(\text{Diff}_\partial(D^{2n}))$ is finite abelian, in fact $B\text{Diff}_\partial(D^{2n})$ is in \mathbf{Fin} . \square

Rationally, the cohomological Serre spectral sequence in the previous proof collapses at E_2 in a range.

Proposition 5.2. *Let $2n \geq 6$, then for $* \leq \frac{g-4}{2}$ we have that*

$$H^*(BW_{g,1} // BD; \mathbb{Q}) \cong \bigoplus_{p+q=*} H^p(\Omega_0^\infty MT\theta; \mathbb{Q}) \otimes H^q(* // BD; \mathbb{Q})$$

Proof. By the homological stability results of Galatius and Randal Williams [GRW14a, GRW14b], we may replace $\Omega_0^\infty MT\theta$ by $B\text{Diff}_\partial(W_{g,1})$ in the relevant range. Thus the statement is a consequence of Leray-Hirsch, Theorem 4.D.1 of [Hat02], once we prove that $B\text{Diff}_\partial(W_{g,1}) \rightarrow BW_{g,1} // BD$ induces a surjection in rational cohomology in the range $* \leq \frac{g-4}{2}$.

Lemma 1.4.1 of [Wei15] proves that as topological monoid, $\text{Emb}_{1/2\partial}^\infty(W_{g,1})$ is weakly equivalent to $\text{Diff}_\partial^*(W_{g,1})$, the topological group of homeomorphisms that are smooth away from a single point $x \in D^{n-1} \subset \partial W_{g,1}$. Using Lemma 4.30 the map $B\text{Diff}_\partial(W_{g,1}) \rightarrow B\text{Top}_\partial(W_{g,1})$ can be factored as

$$B\text{Diff}_\partial(W_{g,1}) \rightarrow BW_{g,1} // BD \rightarrow B\text{Top}_\partial(W_{g,1})$$

In [ERW14], Ebert and Randal-Williams proved that $B\text{Diff}_\partial(W_{g,1}) \rightarrow B\text{Top}_\partial(W_{g,1})$ is surjective on rational cohomology in the range $* \leq \frac{g-4}{2}$, and hence so is $B\text{Diff}_\partial(W_{g,1}) \rightarrow BW_{g,1} // BD$. \square

Theorem 1.1 of [GRW14b] describes $H^*(\Omega_0^\infty MT\theta; \mathbb{Q})$ as the free graded-commutative algebra on classes κ_c of degree $|c| - 2n$, indexed by monomials c in $e, p_{n-1}, p_{n-2}, \dots, p_{\lceil \frac{n+1}{4} \rceil}$ (with degrees given by $|e| = 2n$, $|p_i| = 4i$) of total degree $> 2n$. As $* // BD$ can be delooped further by Remark 4.22 and hence its rational cohomology is graded-commutative with generated by the linear duals of rational homotopy groups, this proposition reduces thus the problem of computing $\pi_*(B\text{Diff}_\partial(D^{2n}); \mathbb{Q})$ to computing $H^*(B\text{Emb}_{1/2\partial}^\infty(W_{g,1}); \mathbb{Q})$.

5.2. Diffeomorphisms of $2n$ -dimensional manifolds. We now prove the first corollaries, about diffeomorphisms of 2-connected manifolds. We single out spheres because the proof is much simpler.

Corollary 5.3. *Let $2n \neq 4$, then $B\text{Diff}(S^{2n})$ is of finite type.*

Proof. We have that $\text{Diff}(S^n) \simeq \text{Diff}_\partial(D^n) \times O(n+1)$, so this follows directly from Theorem 5.1. \square

Corollary 5.4. *Let $2n \neq 4$. Suppose that M is a closed 2-connected oriented smooth $2n$ -dimensional manifold and let $M_1 := M \setminus \text{int}(D^{2n})$. Then we have that*

- (i) $B\text{Diff}_\partial^{\text{id}}(M_1) \in \text{Fin}$,
- (ii) $B\text{Diff}_\partial(M_1) \in \text{HFin}$,
- (iii) $B\text{Diff}^{\text{id}}(M) \in \text{Fin}$,
- (iv) $B\text{Diff}^+(M) \in \text{HFin}$.

Proof. We restrict to $2n \geq 6$, since there is no 2-connected closed surface. Consider the fiber sequence of Corollary 4.35, which loops to a fiber sequence

$$(6) \quad B\text{Diff}_\partial^H(D^{2n}) \rightarrow B\text{Diff}_\partial^{\text{id}}(M_1) \rightarrow \text{BM}^{\text{id}} // \text{BD}^H$$

with H the second inertia group of M . Since $\pi_0(\text{Diff}_\partial(D^n)) \cong \Theta_{n+1}$ is finite, $B\text{Diff}_\partial^H(D^{2n})$ is a finite cover of $B\text{Diff}_\partial(D^{2n})$. Hence by Theorem 5.1 and Lemma 2.5, the fiber of (6) is in Fin . By Proposition 3.8, the base of (6) is in Fin . By parts (ii) and (ii') of Lemma 2.20 and the fact that base is simply-connected, the total space of (6) is in Fin . This proves part (i).

For part (ii), there is a second fiber sequence

$$B\text{Diff}_\partial^{\text{id}}(M_1) \rightarrow B\text{Diff}_\partial(M_1) \rightarrow B\pi_0(\text{Diff}_\partial(M_1))$$

and by Proposition 3.3, the base is in HFin . Applying part (ii) for total spaces of Lemma 2.5 we conclude that $B\text{Diff}_\partial(M_1) \in \text{HFin}$.

For part (iii), we let the identity component of $\text{Diff}(M)$ act on $\text{Emb}^+(D^n, M)$ we get a fiber sequence

$$\text{Emb}^+(D^n, M) \rightarrow B\text{Diff}_\partial^T(M_1) \rightarrow B\text{Diff}^{\text{id}}(M)$$

where $B\text{Diff}_\partial^T(M_1)$ is the classifying space of the subgroup of $\text{Diff}_\partial(M_1)$ given by those components of that become isotopic to the identity when gluing in a disk. This group of components is generated by a Dehn twist along the boundary (i.e. the diffeomorphisms obtained by inserting an element of $\pi_1(SO(n)) \cong \mathbb{Z}/2\mathbb{Z}$ on a collar $[0, 1] \times S^{n-1}$ of the boundary of M_1), and hence is either $\mathbb{Z}/2\mathbb{Z}$ or trivial. See page 302 of [Wal65] for an example. By part (i) of Lemma 2.20 and the fact that the base is simply-connected, we conclude that $B\text{Diff}^{\text{id}}(M) \in \text{Fin}$.

Finally, for part (iv) we use that Theorem 13.3 of [Sul77] proves $B\pi_0(\text{Diff}^+(M)) \in \text{HFin}$ and we can finish the proof by applying part (ii) for bases of Lemma 2.5 to the fiber sequence

$$B\text{Diff}^{\text{id}}(M) \rightarrow B\text{Diff}^+(M) \rightarrow B\pi_0(\text{Diff}^+(M)) \quad \square$$

In [ERW15], Ebert and Randal-Williams studied the higher-dimensional analogue of the Torelli group, $\text{Tor}_\partial(W_{g,1})$, defined as the subgroup of $\text{Diff}_\partial(W_{g,1})$ given by components that act trivially on $H_n(W_{g,1})$. For surfaces of genus ≥ 7 , some rational homology group of the Torelli group are infinite-dimensional [Aki01]. In high dimensions this is not the case:

Corollary 5.5. *Let $2n \geq 6$. Then $B\text{Tor}_\partial(W_{g,1})$ is of homologically finite type.*

Proof. The components of $\mathrm{Tor}_\partial(W_{g,1})$ form a finitely generated nilpotent group by work of Kreck [Kre79]. Such groups have classifying spaces in \mathbf{HFin} by Theorem 2.12 and the corollary then follows from the fiber sequence

$$B\mathrm{Diff}_\partial^{\mathrm{id}}(W_{g,1}) \rightarrow B\mathrm{Tor}_\partial(W_{g,1}) \rightarrow B\pi_0(\mathrm{Tor}_\partial(W_{g,1}))$$

combined with Corollary 5.4 and part (ii) for total spaces of Lemma 2.5. \square

5.3. Diffeomorphisms of the odd-dimensional disk. . The proof in the odd-dimensional case is similar to that in the even-dimensional case, replacing $W_{g,1}$ with the manifold

$$H_g := \natural_g(D^{n+1} \times S^n)$$

where \natural denotes the boundary connected sum. Note that its boundary ∂H_g is given by W_g . We establish some notation.

- Fix a disk $D^{2n} \subset \partial H_g$, so that $\partial H_g \setminus \mathrm{int}(D^{2n}) \cong W_{g,1}$. Recall from Subsection 4.4.2 that $\mathrm{Diff}_{\partial, W_{g,1}}(H_g)$ is the subgroup of $\mathrm{Diff}(H_g)$ consisting of those diffeomorphisms that fix D^{2n} pointwise and $\partial H_g \setminus \mathrm{int}(D^{2n})$ setwise.
- Fix a smaller disk $D' \cong D^{2n} \subset \mathrm{int}(D^{2n})$. Recall that $\mathrm{Emb}_{1/2\partial, W_{g,1}}^\cong(H_g)$ is the subspace of $\mathrm{Emb}(H_g)$ consisting of those self-embeddings that fix $D^{2n} \setminus \mathrm{int}(D')$ pointwise and fix $\partial H_g \setminus \mathrm{int}(D^{2n})$ setwise, and are isotopic through such embeddings to a diffeomorphism.
- Let $\mathrm{Diff}_\partial^{\mathrm{ext}}(W_{g,1})$ be the subgroup of $\mathrm{Diff}_\partial(W_{g,1})$ consisting of those diffeomorphisms of $W_{g,1}$ fixing the boundary that extend to a diffeomorphism of H_g that fixes D^{2n} .

Proposition 3.3 does not apply to the connected components of the last of these groups. Instead we will use work of Kreck and Wall on isotopy classes of diffeomorphisms of handlebodies [Kre79, Wal65]. Wall's results are stated for pseudoisotopy classes, but Cerf proved pseudoisotopy classes coincide with isotopy classes for simply-connected manifolds of dimension ≥ 5 [Cer70].

Lemma 5.6. *If $n \geq 4$, the group $\pi_0(\mathrm{Diff}_\partial^{\mathrm{ext}}(W_{g,1}))$ has a classifying space in \mathbf{HFin} .*

Proof. Kreck gave a complete description of $\pi_0(\mathrm{Diff}_\partial(W_{g,1}))$ up to extensions in terms of two short exact sequences (see e.g. Section 7 of [GRW16]). We will only need one:

$$1 \rightarrow I_{g,1}^n \rightarrow \pi_0(\mathrm{Diff}_\partial(W_{g,1})) \rightarrow \mathrm{Aut}(H_n, \lambda, \alpha) \rightarrow 1$$

Here H_n is the middle-dimensional homology group $H_n(W_{g,1})$ and $\mathrm{Aut}(H_n, \lambda, \alpha)$ is the arithmetic group of automorphisms of the intersection form and its quadratic refinement. The group $I_{g,1}^n$ is finitely generated abelian, and admits a more detailed description which we will not need. Thus $\pi_0(\mathrm{Diff}_\partial(W_{g,1}))$ is an extension of an arithmetic group by a finitely generated nilpotent group, confirming the first part of Proposition 3.3 in this particular case.

Theorem 6 of [Wal65] says we can describe the subgroup of isotopy classes of diffeomorphisms that extend to H_g by replacing $\mathrm{Aut}(H_n, \lambda, \alpha)$ with the subgroup consisting of those elements that preserve $K = \ker(H_n(W_{g,1}) \rightarrow H_n(H_g)) \subset H_n$, and replacing $I_{g,1}^n$ by a subgroup. This is also an extension of an arithmetic group by a finitely generated nilpotent group, so has a classifying space in \mathbf{HFin} . \square

Corollary 5.7. *Let $2n + 1 \neq 5, 7$, then $B\mathrm{Diff}_\partial(D^{2n+1})$ is in \mathbf{Fin} . It is thus in particular of homologically and homotopically finite type.*

Proof. The cases $2n + 1 = 1, 3$ are respectively folklore and [Hat83], so we focus on the case $2n + 1 \geq 9$. Similar to the proof in Theorem 5.1, we start by remarking it suffices to prove that $* // BD \in \text{Fin}$ using the fiber sequence of Corollary 4.37 applied to $M = H_g$ and $A = W_{g,1} \subset \partial H_g$. Its middle term $BH_{g,W_{g,1}} // B\text{Dis}$ is weakly equivalent to $B\text{Emb}_{1/2\partial, W_{g,1}}^{\cong}(H_g)$ and there is a fiber sequence

$$B\text{Emb}_{1/2\partial}^{\cong}(H_g) \rightarrow B\text{Emb}_{1/2\partial, W_{g,1}}^{\cong}(H_g) \rightarrow B\text{Diff}_{\partial}^{\text{ext}}(W_{g,1})$$

The base has a classifying space in ΠFin , using Corollary 5.4 and Lemma 5.6 for the identity component and group of components respectively, and hence in HFin by Lemma 2.15. The fiber has a classifying space in HFin using Theorem 3.2. Hence using part (ii) of Lemma 2.5 we obtain that $BH_{g,W_{g,1}} // BD \in \text{HFin}$. From this point, the argument continues as in the proof of Theorem 5.1 with the results of Botvinnik-Perlmutter [BP15, Per15] replacing those of Galatius-Randal-Williams [GRW14b, GRW14a]. \square

Remark 5.8. The restriction $2n + 1 \neq 5, 7$ is due to the use of a higher version of the Whitney trick by Botvinnik and Perlmutter. The ordinary Whitney trick is limited to dimension ≥ 5 due to an application of transversality to make an immersed Whitney disk embedded. A similar transversality argument is used for the higher version of the Whitney trick, and has a stronger dimension restriction. It may be the case that automorphisms of manifolds of dimension 5 and 7 behave in a qualitatively different manner, cf. Question 1.3.

A similar proof as that for Corollaries 5.3 and 5.4 now gives us the odd-dimensional case.

Corollary 5.9. *Let $2n + 1 \neq 5, 7$, then $B\text{Diff}(S^{2n+1})$ is of finite type.*

Corollary 5.10. *Suppose that M is a closed compact 2-connected $(2n + 1)$ -dimensional oriented smooth manifold and $2n + 1 \neq 5, 7$. Let $M_1 := M \setminus \text{int}(D^{2n+1})$. Then we have that*

- (i) $B\text{Diff}_{\partial}^{\text{id}}(M_1) \in \text{Fin}$,
- (ii) $B\text{Diff}_{\partial}(M_1) \in \text{HFin}$,
- (iii) $B\text{Diff}^{\text{id}}(M) \in \text{Fin}$,
- (iv) $B\text{Diff}^+(M) \in \text{HFin}$.

5.4. Homeomorphisms and PL-homeomorphisms of Euclidean space. Recall that $\text{Top}(n)$ denotes the topological group of homeomorphisms in the compact-open topology and $\text{PL}(n)$ denote the simplicial group of piecewise-linear homeomorphisms. We recall some results about these groups. The first concerns the lower homotopy groups of automorphisms of \mathbb{R}^n , see Section V.5 of [KS77]:

Theorem 5.11 (Kirby-Siebenmann). *Let $n \geq 5$, then we have that $\pi_i(\text{PL}(n)/O(n)) \rightarrow \pi_i(\text{PL}/O)$ is an isomorphism for $i \leq n + 1$ and a surjection for $i = n + 2$. We also have that*

$$\pi_i(\text{PL}/O) \cong \begin{cases} 0 & \text{if } i \leq 4 \\ \Theta_i & \text{otherwise} \end{cases} \quad \pi_i(\text{Top}(n)/\text{PL}(n)) \cong \pi_i(\text{Top}/\text{PL}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } i = 3 \\ 0 & \text{otherwise} \end{cases}$$

The second result concerns smoothing theory for PL-manifolds, e.g. Theorem 4.4 of part I of [BL74].

Theorem 5.12 (Burghleia-Lashof). *We have that $\Theta_n \times B\text{Diff}_{\partial}(D^n) \simeq \Omega^n \text{PL}(n)/O(n)$.*

Corollary 5.13. *Let $n \neq 4, 5, 7$. $B\text{Top}(n)$ and $B\text{PL}(n)$ are in Fin . They are thus in particular of homotopically and homologically finite type.*

Proof. For $n \leq 3$ we have that $O(n) \simeq \text{PL}(n) \simeq \text{Top}(n)$, so we will focus on $n \geq 6$. Since $O(n)$ is homotopically finite type and Θ_i is a finite abelian group, Theorem 5.11 implies that $\pi_i(O(n)/\text{PL}(n))$ is finite for $i \leq n+1$. Theorem 5.12 says $\Theta_n \times B\text{Diff}_\partial(D^n) \simeq \Omega^n \text{PL}(n)/O(n)$. Since $B\text{Diff}_\partial(D^n) \in \text{Fin}$ by Theorem A, $\Omega^n \text{PL}(n)/O(n) \in \text{Fin}$ and thus we also have $\text{PL}(n)/O(n) \in \text{Fin}$.

There is a fiber sequence

$$\text{PL}(n)/O(n) \rightarrow BO(n) \rightarrow B\text{PL}(n)$$

and since $BO(n) \in \text{Fin}$, the long exact sequence for homotopy groups implies $B\text{PL}(n) \in \text{Fin}$. That $B\text{Top}(n) \in \text{Fin}$ then follows from the last part of Theorem 5.11. \square

5.5. Homeomorphisms of $2n$ -dimensional manifolds. Finally, we prove similar results for homeomorphisms and PL-homeomorphisms as we proved for diffeomorphisms.

Corollary 5.14. *Let $n \neq 4, 5, 7$, then $B\text{Top}(S^n)$ and $B\text{PL}(S^n)$ are of finite type.*

Proof. Let $\text{CAT} = \text{Top}$ or PL . Let $\text{Fr}^{\text{CAT}}(TS^n)$ be the bundle over S^n with fiber over $x \in S^n$ given by the CAT -isomorphisms $\mathbb{R}^n \rightarrow T_x S^n$. Acting by a homeomorphism on the tangent microbundle gives a fiber sequence

$$BC\text{AT}_\partial(D^n) \rightarrow BC\text{AT}(S^n) \rightarrow \text{Fr}^{\text{CAT}}(TS^n)$$

By the Alexander trick $\text{CAT}_\partial(D^n) \simeq *$, so $BC\text{AT}(S^n) \simeq \text{Fr}^{\text{CAT}}(TS^n)$. The fiber of $\text{Fr}^{\text{CAT}}(TS^n)$ is equivalent to $\text{CAT}(n)$, so we can prove the result using Corollary 5.13 by applying part (ii) and (ii') of Lemma 2.20 to the fiber sequence

$$\text{CAT}(n) \rightarrow \text{Fr}^{\text{CAT}}(TS^n) \rightarrow S^n$$

\square

For general manifolds, the technique is smoothing theory for embeddings.

Corollary 5.15. *Suppose that M is a closed compact 2-connected n -dimensional smooth manifold, and $n \neq 4, 5, 7$. Let $M_1 := M \setminus \text{int}(D^n)$. If $\text{CAT} = \text{Top}, \text{PL}$ then $BC\text{AT}_\partial(M_1)$ and $BC\text{AT}(M)$ are of homologically finite type. Furthermore, the classifying spaces of their identity components are of homotopically and homologically finite type.*

Proof. We start with the identity component. Let $n \geq 5$ and $\text{CAT} = \text{PL}, \text{Top}$, then Corollary 2 of [Las76] says that

$$\text{hofib}_{\text{id}}(\text{Imm}_{1/2\partial}^{\cong}(M) \rightarrow \text{Imm}_{1/2\partial}^{\text{CAT}}(M)) \simeq \text{hofib}_{\text{id}}(\text{Emb}_{1/2\partial}^{\cong}(M) \rightarrow \text{Emb}_{1/2\partial}^{\text{CAT}}(M))$$

By immersion theory [HP64, Lee69], the left hand side is equivalent to the space of sections $\Gamma(E, M; 1/2\partial M)$ of M of a bundle E with fiber $\text{CAT}(n)/O(n)$, that are equal to a fixed section on half the boundary. As a consequence there is a fiber sequence

$$\Gamma(E, M; 1/2\partial M) \rightarrow \text{Emb}_{1/2\partial}^{\text{Diff}, \text{id}}(M) \rightarrow \text{Emb}_{1/2\partial}^{\text{CAT}, \text{id}}(M)$$

Since by Corollary 5.13, $\text{Top}(n)$ and $\text{PL}(n)$ are homotopically finite type if $n \neq 4, 5, 7$, Lemma 2.21 says that the homotopy groups of a component of $\Gamma(E, M; 1/2\partial M)$ are finitely generated. Furthermore, since the first n homotopy groups of $\text{CAT}(n)/O(n)$ are finite by

Theorem 5.11, $\Gamma(E, M; 1/2\partial M)$ has finitely many components. By Proposition 3.8 we have that $\text{Emb}_{1/2\partial}^{\text{Diff}, \text{id}}(M) \in \Pi\text{Fin}$. We can then use part (i) and (i') of Lemma 2.20 to conclude $\text{Emb}_{1/2\partial}^{\text{CAT}, \text{id}}(M) \in \Pi\text{Fin}$. By Corollary 4.38, $BCAT_{1/2\partial}^{\text{id}}(M_1) \simeq B\text{Emb}_{1/2\partial}^{\text{id}}(M_1)$, so the former is in ΠFin .

Next is the group of connected components. The proof of Theorem 13.3 of [Sul77] only uses surgery theory and that the first n rational homotopy groups of $O(n)$ are finite-dimensional, so his argument also applies to CAT-manifolds. This is the Remark on page 3392 of [Tri95]. As in Proposition 3.3, we have that $B\pi_0(\text{CAT}_{1/2\partial}(M)) \in \text{HFin}$. Applying part (ii) of Lemma 2.5 for total spaces to

$$BCAT_{1/2\partial}^{\text{id}}(M_1) \rightarrow BCAT_{1/2\partial}(M_1) \rightarrow B\pi_0(\text{CAT}_{1/2\partial}(M_1))$$

we conclude that $BCAT_{1/2\partial}(M_1) \in \text{HFin}$.

As in the smooth case, there is a fiber sequence

$$\text{Emb}^{\text{CAT}, +}(D^n, M) \rightarrow BCAT_{\partial}^T(M_1) \rightarrow BCAT^{\text{id}}(M)$$

where $\text{Emb}^{\text{CAT}, +}(D^n, M)$ is the space of orientation-preserving embeddings and $BCAT_{\partial}^T(M_1)$ is the classifying space of the subgroup of $\text{CAT}_{\partial}(M_1)$ on those components that become isotopic to the identity after gluing in a disk. As before, this group of components is either $\mathbb{Z}/2\mathbb{Z}$ or trivial, and we conclude that $BCAT^{\text{id}}(M) \in \Pi\text{Fin}$. We also have $B\pi_0(\text{CAT}(M)) \in \text{HFin}$, which finishes the proof. \square

5.6. Concordance diffeomorphisms and the Wh^{Diff} -spectrum. Finally, we give an application to concordance diffeomorphisms and algebraic K-theory of spaces.

Definition 5.16. Let $C(M)$ be the topological group of diffeomorphisms of $M \times I$ fixing $\partial M \times I$ and $M \times \{0\}$. These are called the *concordance diffeomorphisms*.

Corollary 5.17. *Let $n \geq 8$, then $C(D^n)$ is of finite type.*

Proof. There is a fiber sequence $\text{Diff}_{\partial}(D^{n+1}) \rightarrow C(D^n) \rightarrow \text{Diff}_{\partial}(D^n)$ so the result follows from Theorem 5.1 and Corollary 5.7. \square

Igusa's pseudoisotopy stability theorem [Igu88] says there is a stabilization map $C(M \times I^k) \rightarrow C(M \times I^{k+1})$, which is an equivalence in a range going to ∞ as $k \rightarrow \infty$. Assume for simplicity that M is 1-connected. By Waldhausen's parametrized stable h -cobordism theorem, the space $\text{colim}_{k \rightarrow \infty} BC(M)$ is the infinite loop space $\Omega^{\infty+1}\text{Wh}^{\text{Diff}}(M)$ [WJR13]. The spectrum $\text{Wh}^{\text{Diff}}(M)$ is a summand of $A(M)$: $A(M) \simeq \text{Wh}^{\text{Diff}}(M) \vee \Sigma^{\infty} M_+$. Both $A(M)$ and $\text{Wh}^{\text{Diff}}(M)$ only depend on the homotopy type of M . Since $A(M)$ is connective and satisfies $\pi_0 A(M) = \mathbb{Z}$, from the splitting one deduces that $\pi_i(\text{Wh}^{\text{Diff}}(M)) = 0$ for $i \leq 0$. Thus Corollary 5.17 implies the following result of Dwyer [Dwy80]:

Corollary 5.18. *Both $\text{Wh}^{\text{Diff}}(*)$ and $A(*)$ are of finite type.*

5.7. Homotopy automorphisms and block automorphisms. After the previous sections, two types of automorphism groups of manifolds remain to be discussed: homotopy automorphisms and block automorphisms. Finiteness results for these groups go back to Farrell-Hsiang and Burghlea [FH78, Bur79]. Rationally more precise results were obtained by Berglund and Madsen [BM14]. Since these results can be obtained with relative ease, we

include proofs of them for completeness. We will assume that our manifolds are closed, but we expect this restriction to be unnecessary.

Definition 5.19. Let $\text{haut}(M)$ be the group-like topological monoid of *homotopy automorphisms* of M , i.e. the union of those components of $\text{Map}(M, M)$ that are homotopy equivalences.

Proposition 5.20. *Suppose that M is a closed n -dimensional manifold with finite fundamental group. Then we have that*

- (i) $B\text{haut}^{\text{id}}(M) \in \mathbf{Fin}$,
- (ii) $B\text{haut}(M) \in \mathbf{HFin}$.

Proof. Since the identity component of $\text{haut}(M)$ is a path-connected topological monoid, it suffices to prove it has finitely-generated homotopy groups. By Lemma 2.21, identity component in the space of pointed map $\text{Map}_*(M, M)$ has finitely-generated homotopy groups. It fits into a fiber sequence

$$\text{Map}_*^{\text{id}}(M, M) \rightarrow \text{Map}^{\text{id}}(M, M) \rightarrow M$$

and the result follows from part (ii) and (ii') of Lemma 2.20 using the fact that there is a section $M \rightarrow \text{Map}^{\text{id}}(M, M)$.

We want to apply part (ii) of Lemma 2.5 to the fiber sequence

$$B\text{haut}^{\text{id}}(M) \rightarrow B\text{haut}(M) \rightarrow B\pi_0(\text{haut}(M))$$

and to do so, it suffices to show $B\pi_0(\text{haut}(M)) \in \mathbf{HFin}$. Triantafyllou proved that for a finite CW complex X with finite fundamental group $\pi_0(\text{haut}(X))$ is an arithmetic group [Tri95], and hence we can apply Theorem 2.12. \square

Next we consider block automorphisms. We take $\text{CAT} = \text{Top}, \text{PL}$ or Diff .

Definition 5.21. The group $\widetilde{\text{CAT}}(M)$ of *CAT block automorphisms* of a CAT-manifold M is the simplicial group with k -simplices given by the CAT-isomorphisms

$$f: \Delta^k \times M \rightarrow \Delta^k \times M$$

such that $f(\sigma \times M) = \sigma \times M$ for every face σ of Δ^k .

This group is designed to be studied by surgery. The difference between block automorphisms and homotopy automorphisms is the subject of Quinn's surgery exact sequence [Qui70]: if $n \geq 5$ (or $n = 4$, $\text{CAT} = \text{Top}$ and $\pi_1(M)$ is good [FQ90]) the following is a fiber sequence when restricted to identity components:

$$\text{haut}(M)/\widetilde{\text{CAT}}(M) \rightarrow \text{Map}(M, G/\text{CAT}) \rightarrow \mathbb{L}(M)$$

where $G = \text{colim}_{n \rightarrow \infty} \text{haut}_*(S^n)$ and $\mathbb{L}(-)$ is the quadratic L -theory space.

Proposition 5.22. *Suppose that M is a closed oriented manifold of dimension $n \geq 5$ with finite fundamental group (or a 1-connected topological 4-manifold), then we have that*

- (i) $B\widetilde{\text{CAT}}^{\text{id}}(M) \in \mathbf{Fin}$,
- (ii) $B\widetilde{\text{CAT}}(M) \in \mathbf{HFin}$.

Proof. There is a fiber sequence

$$(7) \quad \text{haut}^{\text{id}}(M)/\widetilde{\text{CAT}}^{\text{id}}(M) \rightarrow \widetilde{BCAT}^{\text{id}}(M) \rightarrow B\text{haut}^{\text{id}}(M)$$

and since we saw in the proof of Proposition 5.20 that the base in (7) is in Fin , using Lemma 2.20 it suffices to prove that the fiber has finitely generated homotopy groups.

This uses the surgery exact sequence. We will first prove that the path components of $\text{Map}(M, G/\text{CAT})$ and $\mathbb{L}(M)$ have finitely generated homotopy groups. Note that $\pi_i(G) \cong \pi_i(\mathbb{S})$ for $i \geq 1$, while $\pi_0(G) = \mathbb{Z}/2\mathbb{Z}$. If $\text{CAT} = \text{Diff}$, we have that $\text{colim}_{n \rightarrow \infty} \text{CAT}(n) \simeq O$ and by Bott periodicity the homotopy groups of O are finitely generated, so that $G/O \in \text{Fin}$. From this and Theorem 5.11, we can conclude that $G/\text{CAT} \in \text{Fin}$ and in fact it has abelian π_1 . By Lemma 2.21 each component of $\text{Map}(M, G/\text{CAT})$ has finitely generated homotopy groups. For the second term, we have that $\pi_i(\mathbb{L}(M)) \cong \pi_{i+n}(\mathbb{L}(\mathbb{Z}[\pi_1(M)]))$, which is finitely generated when $\pi_1(M)$ is finite [HT00].

That $\text{haut}^{\text{id}}(M)/\widetilde{\text{CAT}}^{\text{id}}(M)$ has finitely generated homotopy groups then follows from part (iii) and (iii') of Lemma 2.20, where we use that condition (a) since $\pi_1(\text{Map}(M, G/\text{CAT}))$ is abelian because G/CAT is an infinite loop space [MM79, chapter 6]. We conclude that $\widetilde{BCAT}^{\text{id}}(M) \in \text{Fin}$, which finishes the proof of part (i) of this Proposition.

For part (ii), we intend to apply part (ii) of Lemma 2.5 to the fiber sequence

$$\widetilde{BCAT}^{\text{id}}(M) \rightarrow \widetilde{BCAT}(M) \rightarrow B\pi_0(\widetilde{\text{CAT}}(M))$$

and it thus suffices to prove that $B\pi_0(\widetilde{\text{CAT}}(M)) \in \text{HFin}$. To do this, we remark there is a surjection $\pi_0(\text{CAT}(M)) \rightarrow \pi_0(\widetilde{\text{CAT}}(M))$. Lemma 5.23 says that classifying space of its kernel is in HFin . Triantafyllou proved that $\pi_0(\text{CAT}(M))$ is arithmetic when $\dim M \geq 5$ and $\pi_1(M)$ is finite [Tri95] and hence $B\pi_0(\text{CAT}(M)) \in \text{HFin}$ by Theorem 2.12. Using part (i) of Lemma 2.8, which says that a group with classifying spaces in HFin are closed under quotients, we conclude that $\pi_0(\widetilde{\text{CAT}}(M)) \in \text{HFin}$.

If $n = 4$ and $\pi_1(M) = 0$, then Quinn proved that $\pi_0(\text{Top}(M))$ is arithmetic and pseudoisotopy implies isotopy [Qui86]. \square

Lemma 5.23. *If $n \geq 5$, $\pi_1(M)$ is finite and M is oriented, then the kernel K of the map $\pi_0(\text{CAT}(M)) \rightarrow \pi_0(\widetilde{\text{CAT}}(M))$ has classifying space in HFin .*

Proof. Proposition II.5.1 of [HW73] describes K as the quotient of $\pi_0(C(M))$ by the subgroup $\pi_0(\text{Diff}_{\partial}(M \times I))$, with $C(M)$ as in Definition 5.16. Note that $\pi_0(\text{Diff}_{\partial}(M \times I))$ is abelian, so it suffices to show $B\pi_0(C(M)) \in \text{HFin}$. Theorem 3.1 of [Hat78] gives an exact sequence

$$\begin{aligned} H_0(\pi_1(M); \pi_2(M)[\pi_1(M)]/\pi_2(M)[1]) &\rightarrow \pi_0(C(M)) \\ &\rightarrow \text{Wh}_2(\pi_1(M)) \oplus H_0(\pi_1(M); \mathbb{Z}/2\mathbb{Z}[\pi_1(M)]/\mathbb{Z}/2\mathbb{Z}[1]) \rightarrow 0 \end{aligned}$$

with $\text{Wh}_2(\pi_1(M))$ a quotient of $K_2(\mathbb{Z}[\pi_1(M)])$. If $\pi_1(M)$ is finite, then the universal cover \tilde{M} is a finite CW-complex and thus $\pi_2(M) \cong H_2(\tilde{M})$ is finitely generated. Both $H_0(\pi_1(M); \pi_2(M)[\pi_1(M)]/\pi_2(M)[1])$ and $H_0(\pi_1(M); \mathbb{Z}/2\mathbb{Z}[\pi_1(M)]/\mathbb{Z}/2\mathbb{Z}[1])$ are coinvariants of actions on finitely generated abelian groups, and hence finitely generated abelian. For $\text{Wh}_2(\pi_1(M))$ we use that $K_2(\mathbb{Z}[G])$ is finitely generated abelian if G is finite, by Theorem 1.1.(i) of [Kuk87]. Now use Theorem 2.12 and parts (i) and (ii) of Lemma 2.8. \square

Finally we deduce a result about the difference between block diffeomorphisms and diffeomorphisms, previously known only in the concordance stable range.

Corollary 5.24. *Suppose that M is closed smooth 2-connected manifold of dimension 6 or ≥ 8 . Then we have that $\widehat{\text{CAT}}(M)/\text{CAT}(M)$ is in ΠFin .*

Proof. Under the conditions of the corollary, pseudoisotopy classes coincide with isotopy classes [Cer70, Rou70, BLR75]. Thus the map $\pi_0(\text{CAT}(M)) \rightarrow \pi_0(\widehat{\text{CAT}}(M))$ is an isomorphism and hence there is a fiber sequence

$$\widehat{\text{CAT}}(M)/\text{CAT}(M) \rightarrow \text{BCAT}^{\text{id}}(M) \rightarrow \text{BCAT}^{\text{id}}(M)$$

and from part (iii) and (iii') of Lemma 2.20, Corollaries 5.4, 5.10 and 5.15, and Proposition 5.22 we conclude that $\widehat{\text{CAT}}(M)/\text{CAT}(M) \in \text{Fin}$. \square

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E-mail address: `kupers@math.ku.dk`

INSTITUT FOR MATEMATISKE FAG, KØBENHAVNS UNIVERSITET, UNIVERSITETSPARKEN 5, 2100 KØBENHAVN Ø.